On Left Adjoints Preserving Colimits in HoTT

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Abstract

We examine how the standard proof that left adjoints preserve colimits behaves in the setting of wild categories, a natural setting for synthetic homotopy theory inside homotopy type theory. We prove that the proof may fail for adjunctions between wild categories. Our core contribution, however, is a sufficient condition on the left adjoint for the proof to go through. The condition, called 2-*coherence*, expresses that the naturality structure of the hom-isomorphism commutes with composition of morphisms. We present two useful examples of this condition in action. First, we use it, along with a new version of a known trick for homogeneous types, to show that the suspension functor preserves graph-indexed colimits. Second, we show that every modality, viewed as a functor on coslices of a type universe, is 2-coherent as a left adjoint to the forgetful functor from the subcategory of modal types, thereby proving that this subcategory is cocomplete. Moreover, we have formalized our main results in Agda.

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Supplementary Material *Software (Agda code)*: `https://github.com/PHart3/colimits-agda/tree/lapc` [12]

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1 Introduction

In category theory, a basic and eminently useful fact is that *left adjoints preserve colimits (LAPC)*. We would like to invoke this classical theorem in the categorical setting of synthetic homotopy theory, the axiomatic and usually type-theoretic study of topological spaces with higher-dimensional structure. This would let us produce new universal constructions of spaces from existing ones via purely algebraic methods. For synthetic homotopy theory carried out in homotopy type theory (HoTT), the appropriate categorical setting is that of *wild categories*, the canonical examples of which are type universes. This notion is a type-theoretic interpretation of an $(\infty, 1)$-category that specifies the data of a 1-category but omits higher coherence data. Despite its naivete, this setting is expressive enough to study concepts like (co)limits and adjunctions inside type theory.

We would like to port *LAPC* to adjunctions between wild categories. In particular, we would like to port the "standard" proof, i.e., the proof one would expect to see based on the category theory literature. This, however, turns out to be harder than one might hope as we produce, inside HoTT, an example of such an adjunction for which the proof fails. Nevertheless, we identify a sufficient condition for the proof to go through. Roughly, it expresses that the adjunction data interacts nicely with the left adjoint's (proof-relevant) composition law. With this condition, combined with a higher version a known technique based on homogeneous types, we show that suspension, an example of a left adjoint, preserves (graph-indexed) colimits, which has applications to the theory of acylic types and to homology theory. We also show that every modality (such as truncation), viewed as a functor on coslices of a type universe, satisfies the condition as a left adjoint to the forgetful functor. As a result, the full wild subcategory of modal types inherits colimits from the ambient coslice. Our proof of this fact differs from the one described for truncations by [24, Section 7.4]. Ours is ultimately simpler by placing modalities in the general context of left adjoints.

The sufficient condition for the standard proof and its application to truncations were mentioned briefly in prior work by Hart and Favonia [11, Remark 24]. In the present work, we place these results in a wider context and explain, for the first time, how one proves them. Moreover, we have formalized all our main results in Agda.

1.1 Motivation

To motivate our work, we should review the well-known proofs of $LAPC$ from classical category theory and explain why the one we choose is the right one to port to wild categories. In addition, we should explain why the issue of porting it deserves the HoTT community's attention. We assume the reader is familiar with the basic notions of category theory.

Consider an adjunction $L \dashv R$ between 1-categories $\mathcal{C}$ and $\mathcal{D}$. Let $\mathcal{J}$ be a small 1-category. The classical theorem states that L preserves $\mathcal{J}$-shaped colimits, and it has two well-known proofs. The first requires that $\mathcal{C}$ and $\mathcal{D}$ admit global colimit functors $\mathrm{colim}_{\mathcal{J}} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ and $\mathrm{colim}_{\mathcal{J}} : \mathcal{D}^{\mathcal{J}} \to \mathcal{D}$ [17, Section V.5]. The proof assumes these colimit functors satisfy some coherence conditions that are automatically true for 1-categories (but *not* for wild ones). It proceeds by using the uniqueness of left adjoints to define an isomorphism $\varphi : \mathrm{colim}_{\mathcal{J}} \circ L^{\mathcal{J}} \cong L \circ \mathrm{colim}_{\mathcal{J}}$. By unfolding the units of two composite adjunctions derived from $L \dashv R$ and showing that φ commutes with these units, we can deduce that φ maps the canonical cocone on $\mathrm{colim}_{\mathcal{J}}(L(F))$ to the induced one on $L(\mathrm{colim}_{\mathcal{J}}(F))$ for all diagrams $F : \mathcal{J} \to \mathcal{C}$. Proving that φ commutes in this way tacitly uses coherence conditions on $L \dashv R$ that hold for 1-categories. Also, to conclude that $L(\mathrm{colim}_{\mathcal{J}}(F))$ is colimiting, the proof tacitly uses the pentagon identity of $\mathcal{D}$ (which holds trivially) to transfer the colimiting property of $\mathrm{colim}_{\mathcal{J}}(L(F))$.

Instead of requiring global colimit functors, the second proof starts with a specific colimit $\mathrm{colim}_{\mathcal{J}}(F)$ of a diagram $F : \mathcal{J} \to \mathcal{C}$ and shows that $L(\mathrm{colim}_{\mathcal{J}}(F))$ is a colimit under $L(F)$. Like the first proof, it secretly uses coherence conditions that hold for 1-categories, a point we'll return to. It argues that for all $Y \in \mathsf{Ob}(\mathcal{D})$, the following chain of isomorphisms with $C :\equiv \mathrm{colim}_{\mathcal{J}}(F)$ equals the canonical post-composition map [19, Theorem 4.5.2]:

$$\mathsf{hom}_{\mathcal{D}}(L(C), Y) \cong \mathsf{hom}_{\mathcal{C}}(C, R(Y)) \cong \mathsf{lim}_i(\mathsf{hom}_{\mathcal{C}}(F_i, R(Y))) \cong \mathsf{lim}_i(\mathsf{hom}_{\mathcal{D}}(L(F_i), Y)) \quad (\texttt{iso})$$

This means that the induced cocone on $L(\mathrm{colim}_{\mathcal{J}}(F))$ is indeed colimiting, i.e., L preserves colimits. Besides avoiding global colimit functors, this proof argues in terms of hom-isomorphisms, which are directly supplied by the usual definition of adjunctions between wild categories. This helps us formulate further laws that such adjunctions must satisfy for the proof to work. Finally, it doesn't rely on bicategorical structure of $\mathcal{C}$ or $\mathcal{D}$. Overall, the second proof, which we call the *standard proof*, is better for wild categories.

So, what makes porting the standard proof an interesting problem? The chain of isomorphisms (`iso`) is not hard to replay in wild category theory. Due to the secret coherence conditions, however, proving that it equals the canonical map becomes a problem. In fact, this equality is sometimes false. This problem is surprising at first glance and easy to miss. It indicates a subtle mismatch between the data used to construct the cocone on $L(\mathrm{colim}_{\mathcal{J}}(F))$

and the data used to construct a hom-type adjunction. (In our setting, hom is a family of types, not necessarily sets.) The latter uses only the 0-dimensional data of L, data coming from the underlying directed graph of $\mathcal{C}$. The latter, however, also uses the composiiton law of L, a 1-dimensional datum. This mismatch has strange effects: we can build two naturally isomorphic left adjoints such that the standard proof goes through for one but not the other! One of the chief virtues of this paper is that it puts this issue into the literature and sets the record straight on the approach to $LAPC$ that was expected to work.

1.2 Contributions

In this section, we explain the contributions of the paper and its organization. We start by outlining the heart of the paper: a coherence condition on the data of a (hom-type) adjunction between wild categories that guarantees the left adjoint preserves colimits. Afterward, we outline two applications of this coherence condition in synthetic homotopy theory. We provide links to corresponding formal proofs in Agda throughout the paper.

1.2.1 2-coherent left adjoints (Section 5)

Let $\mathcal{C}$ and $\mathcal{D}$ be wild categories. (We review the relevant concepts of wild category theory in Section 4.) Our work centers on a new notion of coherence for adjunctions between $\mathcal{C}$ and $\mathcal{D}$. Such an adjunction consists of functors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ together with a family of type equivalences $\psi : \prod_{X:\mathsf{Ob}(\mathcal{D})} \prod_{A:\mathsf{Ob}(\mathcal{C})} \mathsf{hom}_{\mathcal{D}}(L(A), X) \xrightarrow{\simeq} \mathsf{hom}_{\mathcal{C}}(A, R(X))$ and witnesses nat_{cod} and nat_{dom} of the naturality of ψ in X and in A. The main point of this paper is that, despite being a direct translation of the classical notion, this definition is not coherent enough, due to the proof-relevance of the hom types. Indeed, it doesn't let us prove left adjoints preserve colimits, the defining property of left adjoints between locally presentable categories. To solve this problem, we introduce the following coherence condition.

Given an adjunction between $\mathcal{C}$ and $\mathcal{D}$, we say that L is 2-*coherent* if for all suitable morphisms h_1, h_2, and h_3, the identity $\psi(h_1) \circ h_2 \circ h_3 = \psi(h_1 \circ L(h_2 \circ h_3))$ obtained by applying nat_{dom} multiple times equals the identity obtained by applying the composition law $L_\circ$ of L. This nice interaction between nat_{dom} and $L_\circ$ holds automatically for 1-categories, in which case these two equalities are proof-irrelevant. Also, it holds for adjunctions between $(\infty, 1)$-categories by virtue of the infinite tower of coherence data they encode. For wild categories, however, adjunctions may fail to satisfy it.

We prove that 2-coherent left adjoints preserve colimits. The proof proceeds entirely by algebraic manipulation of the adjunction data. Conceptually, it is quite direct as it relies on just three foundational ingredients of HoTT: the naturality of homotopies, the triangle identity of equivalences, and the structure identity principle.

During the proof, we must take care to eliminate the data witnessing that ψ is an equivalence. Otherwise, we would need to include this data in the 2-coherence condition, which would make it far less tractable. Indeed, the condition we arrive at is relatively simple to check and thus quite useful in practice. We just need to know how to compute nat_{dom} and $L_\circ$. In Sections 1.2.2 and 1.2.3, we outline natural examples of left adjoints along with methods for proving that they are 2-coherent.

1.2.2 Suspension preserves colimits (Section 6)

The suspension endofunctor $\Sigma : \mathcal{U}_* \to \mathcal{U}_*$ on the wild category of pointed types is critical to synthetic homotopy theory. For a while, it has been known that Σ is left adjoint to the loop space functor Ω in HoTT. Its preservation of colimits has been expected to follow from *LAPC* in the usual way. Yet, this paper shows that the usual way requires a coherence between the adjunction $\Sigma \dashv \Omega$ and Σ's composition law, which makes the proof trickier than expected.

We verify that this coherence holds, i.e., that Σ is 2-coherent, thereby verifying that Σ preserves colimits. In this case, the final part of the proof of 2-coherence is infeasible to perform directly. Instead, we get it for free by proving a new, higher version of Cavallo's trick for homogeneous types, which include all loop spaces.

This infeasibility highlights a major difference between our type system (Book HoTT) and cubical type theory [25]. Unlike Book HoTT, cubical supports *definitional* β-rules for path constructors in higher inductive types (HITs), such as suspension types. Such support greatly simplifies the proof in question as it erases many postulated equalities that we must carry around. Although constructions with HITs tend to be much harder in Book HoTT [18], they are still valuable. Indeed, Book HoTT is a smaller system in the sense that it admits a shallow embedding into cubical. More importantly, it has better established semantics: It has models in all $(\infty, 1)$-toposes [16, 22], whereas it's not known whether the type theory underlying Cubical Agda has a model Quillen equivalent to the category of topological spaces.

Inside HoTT, the fact that Σ preserves colimits has useful consequences. It implies that the pointed acyclic types [3] are closed under colimits in $\mathcal{U}_*$. Moreover, it puts on firm footing a key step of the construction of stable homotopy as a homology theory [10, Corollary 2.4].

1.2.3 Colimits of modal types (Section 7)

Modalities are functors $\bigcirc : \mathcal{U} \to \mathcal{U}_\bigcirc$ on a type universe $\mathcal{U}$ that arise as reflectors into well-behaved subuniverses $\mathcal{U}_\bigcirc$ of $\mathcal{U}$ [21]. Although they are well-studied in HoTT [6, 7], their interaction with colimits could be explained better. Since reflectors are by definition left adjoints, we should expect that modalities preserve colimits. In the case of pushouts, this property already has a proof for n-truncations $\|-\|_n$ [24, Section 7.4], which are examples of modalities, and it should extend easily to all graph-indexed colimits. The problem is that this proof, which we call the *Book proof*, relies on the particular computational behavior of the composition law of $\|-\|_n$. Thus, it doesn't generalize to arbitrary left adjoints.

We offer a different proof of the same property by showing that every modality $\bigcirc$ is a 2-coherent left adjoint. By fitting into the general framework of *LAPC*, our proof gets rid of the ad-hoc steps required by the Book proof. As a result, ours amounts to an easy application of the induction principle of $\bigcirc$. Moreover, we argue that ours matches the usual classical proof for reflective subcategories because one would normally view colimit preservation here as a special case of *LAPC*.[1] Hence our proof has some advantages over the Book proof.

In fact, given a modality $\bigcirc$, we prove the more general fact that the induced functor $\bigcirc^A : A/\mathcal{U} \to (A/\mathcal{U})_\bigcirc$ on an arbitrary coslice of $\mathcal{U}$ is 2-coherent. (We recover the original case when $A \equiv \mathbf{0}$.) Previously, Hart and Favonia used that $\|-\|_n^A$ is 2-coherent in order to construct colimits of higher groups [11, Section 7]. Here we offer a concise proof that is formulated for all modalities. Finally, we use the colimit preservation of $\bigcirc^A$ together with some general results about *wild bicategories* to build colimits in $(A/\mathcal{U})_\bigcirc$ from those in $A/\mathcal{U}$. (The latter are explicitly constructed by [11, Section 5].)

[1] This goes against [24, Section 7.7]'s claim that the Book proof matches the usual classical one.

2 Additional related work

2.1 Wild category theory

Our work establishes a fundamental property of well-behaved adjunctions between wild categories. It thus fits neatly into the rich theory of wild categories developed in HoTT [8, 13]. In particular, we continue the study of adding higher coherence data to wild-categorical notions. So far, this study has focused on adding conditions internalizing the axioms of a $(2,1)$-category to the wild category itself [4, 5, 11]. The literature calls the resulting concept *wild* 2*-precategory*, 2*-coherent wild category*, or *wild bicategory*. Many natural examples of wild categories carry such structure, including coslices $A/\mathcal{U}$ of a (type) universe, which are important to our work. In fact, we rely on the bicategorical structure of $A/\mathcal{U}$ in Section 7.

In this paper, we focus on a different aspect of 2-coherence: adding it to data *between* wild categories. The ability to isolate which data we make 2-coherent is a virtue of wild category theory: it offers a fine-grained understanding of the effects of higher coherences.

2.2 Homogeneous types

For the application to the suspension functor (Section 6), we build on the theory of *homogeneous types* in HoTT [2, Section 2]. These are pointed types that are independent of basepoint in a strong sense. The key feature of such types is that proving identities about pointed maps into them is considerably easier than about arbitrary pointed maps. This feature, known as *Cavallo's trick*, can make normally intractable computations in higher path algebra tractable. For example, Axel Ljungström adapted Cavallo's trick to show that the smash product forms a symmetric monoidal product on the wild category of pointed types [15]. We provide a different adaptation to handle the 2-coherence condition for the suspension functor.

3 Background on type theory

We review some basic constructions in HoTT that are important for our work. We assume the reader is familiar with Martin-Löf type theory (MLTT), the core type system of HoTT, in the style of [24]. Notably, MLTT is sufficient for the core of our work: all of Section 5 is carried out in MLTT (with function extensionality). For Sections 6 and 7, we postulate a simple class of HITs, pushout types [24, Section 6.8]. In particular, Section 6 focuses on *suspensions*, a kind of pushout.

3.1 Type system

We review three constructions in our type system. The first is the function $\mathsf{ap} : (x = y) \to (f(x) = f(y))$ defined by path induction for all functions $f : X \to Y$ and $x, y : X$. (We use $=$ for the identity type and $\equiv$ for definitional equality.) If we view X as an ∞-groupoid, then ap is the action of f on morphisms of X, thereby exhibiting f as a functor. A key property of ap is the following naturality law.

▶ **Lemma 1** (Homotopy naturality). *Let $f, g : X \to Y$. For all $x, y : X$, $p : x = y$, and*

$H : f \sim g$, we have a commuting square of identities

$$\begin{tikzcd} f(x) \arrow[r, equal, "H(x)"] \arrow[d, equal, "\mathsf{ap}_f(p)"'] & g(x) \arrow[d, equal, "\mathsf{ap}_g(p)"] \\ f(y) \arrow[r, equal, "H(y)"'] & g(y) \end{tikzcd}$$

Here, $f_1 \sim f_2 :\equiv \prod_{x:X} f_1(x) = f_2(x)$ for any dependent functions $f_1, f_2 : \prod_{x:X} Y(x)$, called the type of *homotopies* between f_1 and f_2. If $f_1 \sim f_2$, we say that f_1 and f_2 are *homotopic*.

The second is the notion of *half-adjoint equivalence*. Let $f : X \to Y$ be a function. We say that f is a *half-adjoint equivalence*, or just *equivalence*, if it has a function $g : Y \to X$, homotopies $\eta_f : g \circ f \sim \mathsf{id}_X$ and $\epsilon_f : f \circ g \sim \mathsf{id}_Y$, and a triangle identity $\tau_f(x) : \mathsf{ap}_f(\eta_f(x)) = \epsilon(f(x))$ for every $x : X$. We may denote g by f^{-1}. We use $\simeq$ to refer to equivalences. A function is an equivalence if and only if it is *bi-invertible*, i.e., has a right inverse s and a left inverse r [24, Corollary 4.3.3].

The third is the *transport* function $\mathsf{transp}^Y : \prod_{x,y:X} \prod_{p:x=y} Y(x) \to Y(y)$ for any type family Y over X. This notion gives us a dependent version of ap: If $f : \prod_{x:X} Y(x)$, then we have a function $\mathsf{apd}_f : \prod_{x,y:X} \prod_{p:x=y} \mathsf{transp}^Y(p, f(x)) = f(y)$. As a result, we can generalize Lemma 1 as follows: For all $f, g : X \to Y$, $x, y : X$, $p : x = y$, and $q : f(x) = g(x)$, we have a path $\mathsf{ap}_f(p) \cdot \mathsf{transp}^{f \sim g}(p, q) = q \cdot \mathsf{ap}_g(p)$. The transport function is essential for stating the induction principle of HITs, such as suspensions.

3.2 Suspensions

For all functions $f : X \to Y$, the *cofiber of* f is the pushout of the span $\mathbf{1} \leftarrow X \xrightarrow{f} Y$. Let X be a type. The *suspension* $\Sigma(X)$ *of* X is the cofiber of $X \to \mathbf{1}$. Explicitly, it is the pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathsf{glue}"] & \mathbf{1} \arrow[d, "\mathsf{inr}"] \\ \mathbf{1} \arrow[r, "\mathsf{inl}"'] & \Sigma(X) \end{tikzcd}$$

where $\mathsf{glue} : \prod_{x:X} \mathsf{inl}(*) = \mathsf{inr}(*)$. We denote $\mathsf{inl}(*)$ and $\mathsf{inr}(*)$ by N and S, respectively, and consider N the basepoint of $\Sigma(X)$. The induction principle for $\Sigma(X)$ states that for every type family E over $\Sigma(X)$ with elements

$$t_{\mathsf{N}} : E(\mathsf{N}) \qquad T : \prod_{x:X} \mathsf{transp}^E(\mathsf{glue}(x), t_{\mathsf{N}}) = t_{\mathsf{S}}$$
$$t_{\mathsf{S}} : E(\mathsf{S})$$

we have a function $\mathsf{ind}(E, t_{\mathsf{N}}, t_{\mathsf{S}}, T) : \prod_{z:\Sigma(X)} E(z)$ that satisfies the definitional equalities

$$\mathsf{ind}(E, t_{\mathsf{N}}, t_{\mathsf{S}}, T)(\mathsf{N}) \equiv t_{\mathsf{N}} \qquad \mathsf{ind}(E, t_{\mathsf{N}}, t_{\mathsf{S}}, T)(\mathsf{S}) \equiv t_{\mathsf{S}}$$

and is equipped with an identity $\rho_{\mathsf{ind}(E,t_{\mathsf{N}},t_{\mathsf{S}},T)}(x) : \mathsf{apd}_{\mathsf{ind}(E,t_{\mathsf{N}},t_{\mathsf{S}},T)}(\mathsf{glue}(x)) = T(x)$. In the non-dependent case, this principle is called the *recursion principle*.

Let $(X, x_0), (Y, y_0) : \mathcal{U}_*$ be pointed types in a universe $\mathcal{U}$ and $(f, f_0) : (X, x_0) \to_* (Y, y_0)$ be a pointed map. We have a pointed map $\Sigma(f, f_0) : \Sigma(X, x_0) \to_* \Sigma(Y, y_0)$ defined by recursion on $\Sigma(X)$, which trivially preserves the basepoint. Note that $\rho_{\Sigma(f,f_0)}(x) : \mathsf{ap}_{\Sigma(f,f_0)}(\mathsf{glue}(x)) = \mathsf{glue}(f(x))$ for each $x : X$.

4 Wild category theory

In this section, we record essential concepts and constructions in wild category theory, including adjunctions. The reader will notice that the basic definitions are naive translations of their classical counterparts.

The key distinction between wild categories and the categories of [24, Section 9.1] is that the latter have hom types that behave as *sets*, i.e., have trivial identity types, so that all higher coherences between morphisms in them hold trivially. By contrast, wild categories simply ignore such coherences. In synthetic homotopy theory, many wild categories have hom types that are not sets, so developing the theory of wild categories is worthwhile.

▶ **Definition 2.** *A* wild category (relative to universes $\mathcal{U}$ and $\mathcal{V}$) *is a tuple consisting of a type* $\mathsf{Ob} : \mathcal{U}$ *of objects, a family* $\mathsf{hom} : \mathsf{Ob} \to \mathsf{Ob} \to \mathcal{V}$ *of hom types, identity morphisms* id*, composition* $\circ$*, left and right unit laws for* $\circ$*, and an associativity law* assoc *for* $\circ$*.*

▶ **Example 3.** Let A be a type. The wild category $A/\mathcal{U}$ has objects $\sum_{X:\mathcal{U}} A \to X$ and morphisms $X \to_A Y :\equiv \sum_{k:\mathsf{pr}_1(X)\to\mathsf{pr}_1(Y)} k \circ \mathsf{pr}_2(X) \sim \mathsf{pr}_2(Y)$. Composition and associativity are defined easily via path induction (and the structure identity principle for $\to_A$). The wild category of pointed types $\mathcal{U}_*$, which is isomorphic to $\mathbf{1}/\mathcal{U}$, has a similar structure.

The following notion generalizes the univalence axiom [24, Axiom 2.10.3] and can significantly simplify proofs. We will invoke it in the proof of Corollary 23.

▶ **Definition 4.** *A wild category* $\mathcal{C}$ *is* univalent *if for all* $A, B : \mathsf{Ob}(\mathcal{C})$*, the canonical function* $(A = B) \to (A \simeq_{\mathcal{C}} B)$ *is an equivalence. Here, elements of the right-hand type are* equivalences in $\mathcal{C}$*, defined as bi-invertible morphisms.*

▶ **Definition 5.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be wild categories.*

1. *A* functor $F : \mathcal{C} \to \mathcal{D}$ *from* $\mathcal{C}$ *to* $\mathcal{D}$ *consists of a function* $F_0 : \mathsf{Ob}(\mathcal{C}) \to \mathsf{Ob}(\mathcal{D})$ *and an action on morphisms* $F_1 : \mathsf{hom}_{\mathcal{C}}(X,Y) \to \mathsf{hom}_{\mathcal{D}}(F_0(X), F_0(Y))$ *along with a composition law* $F_\circ : F_1(g) \circ F_1(f) = F_1(g \circ f)$ *and an identity law* $F_{\mathsf{id}} : \mathsf{id}_{F_0(X)} = F_1(\mathsf{id}_X)$*.*
We may refer to F_0 *or* F_1 *by just* F*. We call* F_0 *and* F_1 *the* 0-dimensional *data of* F*. We call* $F_\circ$ *and* F_{id} *its* 1-dimensional *data.*
2. *Let* $F, G : \mathcal{C} \to D$ *be functors. A* natural transformation $\tau : F \Rightarrow G$ *from* F *to* G *consists of functions* $\tau_0 : \prod_{X:\mathsf{Ob}(\mathcal{C})} \mathsf{hom}_{\mathcal{D}}(F(X), G(X))$ *and* $\tau_1 : \prod_{X,Y:\mathsf{Ob}(\mathcal{C})} \prod_{f:\mathsf{hom}_{\mathcal{C}}(X,Y)} G(f) \circ \tau_0(X) = \tau_0(Y) \circ F(f)$*. (If* $\mathcal{D} \equiv \mathcal{U}$*, we may tacitly use the equivalent type* $G(f) \circ \tau_0(X) \sim \tau_0(Y) \circ F(f)$ *instead.) We say* τ *is a* natural isomorphism *if each* $\tau_0(X)$ *is an equivalence.*

▶ **Definition 6** (Adjunction). *Let* $L : \mathcal{C} \to \mathcal{D}$ *and* $R : \mathcal{D} \to \mathcal{C}$ *be functors of wild categories. An adjunction* $L \dashv R$ *is a family of equivalences* $\psi : \mathsf{hom}_{\mathcal{D}}(L(A), X) \simeq \mathsf{hom}_{\mathcal{C}}(A, R(X))$ *equipped with functions witnessing that* ψ *is natural in* X *and* A*, respectively:*

$$\mathsf{nat}_{cod} \;:\; \prod_{A:\mathsf{Ob}(\mathcal{C})} \prod_{X,Y:\mathsf{Ob}(\mathcal{D})} \prod_{g:\mathsf{hom}_{\mathcal{D}}(X,Y)} \prod_{h:\mathsf{hom}_{\mathcal{D}}(L(A),X)} R(g) \circ \psi(h) = \psi(g \circ h)$$

$$\mathsf{nat}_{dom} \;:\; \prod_{Y:\mathsf{Ob}(\mathcal{D})} \prod_{A,B:\mathsf{Ob}(\mathcal{C})} \prod_{f:\mathsf{hom}_{\mathcal{C}}(A,B)} \prod_{h:\mathsf{hom}_{\mathcal{D}}(L(B),Y)} \psi(h) \circ f = \psi(h \circ L(f))$$

For each adjunction $L \dashv R$, we also have naturality squares

$$\begin{tikzcd} \mathsf{hom}_{\mathcal{C}}(A, R(X)) \arrow[r, "R(g)\circ -"] \arrow[d, "\psi^{-1}"'] \arrow[dr, phantom, "\widetilde{\mathsf{nat}}_{cod}(g)"] & \mathsf{hom}_{\mathcal{C}}(A, R(Y)) \arrow[d, "\psi^{-1}"] \\ \mathsf{hom}_{\mathcal{D}}(L(A), X) \arrow[r, "g\circ -"'] & \mathsf{hom}_{\mathcal{D}}(L(A), Y) \end{tikzcd} \qquad \begin{tikzcd} \mathsf{hom}_{\mathcal{C}}(B, R(Y)) \arrow[r, "-\circ f"] \arrow[d, "\psi^{-1}"'] \arrow[dr, phantom, "\widetilde{\mathsf{nat}}_{dom}(f)"] & \mathsf{hom}_{\mathcal{C}}(A, R(Y)) \arrow[d, "\psi^{-1}"] \\ \mathsf{hom}_{\mathcal{D}}(L(B), Y) \arrow[r, "-\circ L(f)"'] & \mathsf{hom}_{\mathcal{D}}(L(A), Y) \end{tikzcd}$$

Here, the right-hand square is defined by the commuting square

$$
\begin{tikzcd}
\psi^{-1}(h) \circ L(f) \arrow[r, equal, "\widetilde{\mathsf{nat}}_{dom}(f,h)"] \arrow[d, equal, "\eta_\psi(\psi^{-1}(h)\circ L(f))"'] & \psi^{-1}(h \circ f) \arrow[d, equal, "\mathsf{ap}_{\psi^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon_\psi(h)))"] \\
\psi^{-1}(\psi(\psi^{-1}(h) \circ L(f))) \arrow[r, equal, "\mathsf{ap}_{\psi^{-1}}(\mathsf{nat}_{dom}(f,\psi^{-1}(h)))"'] & \psi^{-1}(\psi(\psi^{-1}(h)) \circ f)
\end{tikzcd}
$$

where η_ψ and ϵ_ψ come from the equivalence data of ψ. The lefthand square is defined similarly.

▶ Remark. Our results will make no use of F_{id} or nat_{cod}, so we could have omitted them.

In particular, $\left(\psi^{-1}, \widetilde{\mathsf{nat}}_{dom}\right)$ is a natural isomorphism $\mathsf{hom}_{\mathcal{C}}(-, R(Y)) \Rightarrow \mathsf{hom}_{\mathcal{D}}(L(-), Y)$ for each $Y : \mathsf{Ob}(\mathcal{D})$. The terms nat_{dom} and $\widetilde{\mathsf{nat}}_{dom}$ are related by the following *exchange law*.

▶ **Lemma 7.** *Let* $(\psi, \mathsf{nat}_{cod}, \mathsf{nat}_{dom}) : L \dashv R$*. For all* $f : \mathsf{hom}_{\mathcal{C}}(A, B)$ *and* $v : \mathsf{hom}_{\mathcal{D}}(L(B), Y)$*, we have a commuting square*

$$
\begin{tikzcd}
\psi^{-1}(\psi(v)) \circ L(f) \arrow[r, equal, "\widetilde{\mathsf{nat}}_{dom}(f,\psi(v))"] \arrow[d, equal, "\mathsf{ap}_{-\circ L(f)}(\eta_\psi(v))"'] \arrow[dr, phantom, "\mathsf{exch}(f,v)"] & \psi^{-1}(\psi(v) \circ f) \arrow[d, equal, "\mathsf{ap}_{\psi^{-1}}(\mathsf{nat}_{dom}(f,v))"] \\
v \circ L(f) \arrow[r, equal, "\eta_\psi(v\circ L(f))"'] & \psi^{-1}(\psi(v \circ L(f)))
\end{tikzcd}
$$

Proof. Define $\mathsf{exch}(f, v)$ as the chain of paths

$$
\begin{gathered}
\widetilde{\mathsf{nat}}_{dom}(f, \psi(v)) \\
\Big\| \text{ by definition} \\
\eta_\psi(\psi^{-1}(\psi(v)) \circ L(f))^{-1} \cdot \mathsf{ap}_{\psi^{-1}}(\mathsf{nat}_{dom}(f, \psi^{-1}(\psi(v))))^{-1} \cdot \mathsf{ap}_{\psi^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon_\psi(\psi(v)))) \\
\Big\| \text{ via homotopy naturality of } \mathsf{nat}_{dom}(f,-) \\
\eta_\psi(\psi^{-1}(\psi(v)) \circ L(f))^{-1} \cdot \mathsf{ap}_{\psi^{-1}}(\mathsf{ap}_{-\circ f}(\mathsf{ap}_\psi(\eta_\psi(v))) \cdot \mathsf{nat}_{dom}(f, v) \cdot \mathsf{ap}_\psi(\mathsf{ap}_{-\circ L(f)}(\eta_\psi(v)))^{-1})^{-1} \cdot \mathsf{ap}_{\psi^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon_\psi(\psi(v)))) \\
\Big\| \text{ via homotopy naturality of } \eta_\psi \\
\mathsf{ap}_{-\circ L(f)}(\eta_\psi(v)) \cdot \eta_\psi(v \circ L(f))^{-1} \cdot \mathsf{ap}_{\psi^{-1}}(\mathsf{nat}_{dom}(f, v))^{-1} \cdot \mathsf{ap}_{\psi^{-1}}(\mathsf{ap}_{-\circ f}(\mathsf{ap}_\psi(\eta_\psi(v))))^{-1} \cdot \mathsf{ap}_{\psi^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon_\psi(\psi(v)))) \\
\Big\| \text{ via the triangle identity for } \psi \\
\mathsf{ap}_{-\circ L(f)}(\eta_\psi(v)) \cdot \eta_\psi(v \circ L(f))^{-1} \cdot \mathsf{ap}_{\psi^{-1}}(\mathsf{nat}_{dom}(f, v))^{-1}
\end{gathered}
$$

◀

Limits

Since we are studying colimits, we need to discuss cocones under diagrams. In HoTT, we have a concrete description of limits in the wild category of types that offers a useful way of representing cocones in wild categories. To avoid an infinite tower of coherence conditions (an unsolved problem in HoTT), we only consider diagrams over *graphs*, which are type-theoretic versions of free categories [11, Section 3.2]. Let $\mathcal{U}$ be a universe. A *graph* Γ is a pair (Γ_0, Γ_1) consisting of a type $\Gamma_0 : \mathcal{U}$ of vertices and a family $\Gamma_1 : \Gamma_0 \to \Gamma_0 \to \mathcal{U}$ of edges. Given a wild category $\mathcal{C}$, a Γ*-shaped diagram* F *in* $\mathcal{C}$ is a pair (F_0, F_1) consisting of a function $F_0 : \Gamma_0 \to \mathsf{Ob}(\mathcal{C})$ and a family of maps $F_1 : \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} \mathsf{hom}_{\mathcal{C}}(F_0(i), F_0(j))$. We may write F for F_0 and F_1. A natural transformation between two diagrams is defined similarly to one between two functors.

Let Γ be a graph. For every $\mathcal{U}$-valued diagram F over Γ, the *(standard) limit of* F [1, Definition 4.2.7] is the type $\mathsf{lim}(F) :\equiv \sum_{\delta:\prod_{i:\Gamma_0} F_i} \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} F_{i,j,g}(\delta_i) = \delta_j$. The limit is functorial in F. The action on maps sends $\tau : F \Rightarrow G$ to the function $\mathsf{lim}(\tau) : \mathsf{lim}(F) \to \mathsf{lim}(G)$ defined by $(\delta, D) \mapsto \Big(\lambda i.\tau_0(i,\delta_i), \lambda i \lambda j \lambda g.\tau_1(i,j,g,\delta_i) \cdot \mathsf{ap}_{\tau_0(j)}(D_{i,j,g})\Big)$. Let $\mathcal{C}$ be a wild category. For every $\mathcal{C}$-valued diagram F over Γ and every $C : \mathsf{Ob}(\mathcal{C})$, we have the diagram $\mathsf{hom}_{\mathcal{C}}(F_j, C) \xrightarrow{-\circ F_{i,j,g}} \mathsf{hom}_{\mathcal{C}}(F_i, C)$ over Γ^{op}, the *opposite graph of* Γ, similar to the opposite category. We define the type of *cocones under* F *on* C as $\mathsf{lim}_{i:\Gamma^{\mathrm{op}}}(\mathsf{hom}_{\mathcal{C}}(F_i, C))$.

▶ **Lemma 8.** *Let Γ be a graph and let F and G be Γ-shaped diagrams in $\mathcal{U}$. If $\tau : F \Rightarrow G$ is a natural isomorphism, then $\mathsf{lim}(\tau) : \mathsf{lim}(F) \to \mathsf{lim}(G)$ is an equivalence.*

Next, we state the *structure identity principle (SIP)* for lim. The SIP is a general lemma [20, Theorem 11.6.2] characterizing identity types of Σ-types. We will need the SIP for limits in order to port the proof of *LAPC*.

▶ **Lemma 9.** *Let F be a Γ-shaped diagram in $\mathcal{U}$. Let $e_1 :\equiv (\delta_1, D_1), e_2 :\equiv (\delta_2, D_2) : \mathsf{lim}(F)$. Then $e_1 = e_2$ is equivalent to the type of $Q : \delta_1 \sim \delta_2$ equipped with commuting squares*

$$\begin{tikzcd}
F_{i,j,g}(\delta_1(i)) \arrow[r, equal, "D_1(i,j,g)"] \arrow[d, equal, "\mathsf{ap}_{F_{i,j,g}}(Q_i)"'] & \delta_1(j) \arrow[d, equal, "Q_j"] \\
F_{i,j,g}(\delta_2(i)) \arrow[r, equal, "D_2(i,j,g)"'] & \delta_2(j)
\end{tikzcd}$$

5 Porting the proof of *LAPC*

This section is the core of the paper. We find a sufficient, practically useful condition for the standard proof of *LAPC* to work for wild categories. Informally, the condition states that nat_{dom} interacts nicely with the composition law of the left adjoint.

Let $\mathcal{C}$ be a wild category. Let Γ be a graph and $F : \Gamma \to \mathcal{C}$ be a $\mathcal{C}$-valued diagram over Γ. Consider a cocone $\mathcal{K} :\equiv (C, r, K)$ under F, where $r_i : F_i \to C$ for each $i : \Gamma_0$ and $K_{i,j,g} : r_j \circ F_{i,j,g} = r_i$ for all $i, j : \Gamma_0$ and $g : \Gamma_1(i,j)$.

▶ **Definition 10** ([12, is-colim]). *We say that $\mathcal{K}$ is* colimiting *if for every $X : \mathsf{Ob}(C)$, the following post-composition map is an equivalence:*

$$\begin{aligned}
\mathsf{postcomp}_{\mathcal{K}}(X) \ &: \ \mathsf{hom}_{\mathcal{C}}(C, X) \to \mathsf{lim}_{i:\Gamma^{\mathrm{op}}}(\mathsf{hom}_{\mathcal{C}}(F_i, X)) \\
\mathsf{postcomp}_{\mathcal{K}}(X, f) \ &:\equiv \ \big(\lambda i.f \circ r_i, \lambda j \lambda i \lambda g.\, \mathsf{assoc}(f, r_j, F_{i,j,g}) \cdot \mathsf{ap}_{f\circ -}(K_{i,j,g})\big)
\end{aligned}$$

Definition 10 expresses that for every cocone $\mathcal{K}'$ under F, there is a unique cocone morphism $\mathcal{K} \to \mathcal{K}'$. Let $\mathcal{D}$ be a wild category and $L : \mathcal{C} \to \mathcal{D}$ be a functor. We have an induced diagram $L(F)$ and an induced cocone $L(\mathcal{K})$ under $L(F)$:

$$\begin{tikzcd}
L(F_i) \arrow[rr, "L(F_{i,j,g})"] \arrow[dr, "L(r_i)"'] & {} \arrow[d, phantom, "L(K_{i,j,g})"] & L(F_j) \arrow[dl, "L(r_j)"] \\
& L(C) &
\end{tikzcd} \qquad (L(K_{i,j,g}) :\equiv L_\circ(r_j, F_{i,j,g}) \cdot \mathsf{ap}_L(K_{i,j,g}))$$

Suppose that $\mathcal{K}$ is colimiting and that we have an adjunction $(\psi, \mathsf{nat}_{cod}, \mathsf{nat}_{dom}) : L \dashv R$. We wish to replay the standard classical proof by showing the chain of isomorphisms (`iso`) equals the post-composition map. Let us preview this argument. Let ζ denote the composite of the isomorphisms. By function extensionality, it suffices to show ζ and post-composition are equal on h for every $h : \mathsf{hom}_{\mathcal{D}}(L(C), Y)$. For each $i : \Gamma_0$, we can build an equality $Q_i : \psi^{-1}(\psi(h) \circ r_i) = h \circ L(r_i)$ from nat_{dom} and the equivalence data for ψ. By the SIP for lim (Lemma 9), it suffices to prove the following equality for all $i, j : \Gamma_0$ and $g : \Gamma_1(i, j)$:

$$\mathsf{pr}_2(\zeta(h))(j, i, g) \cdot Q_i \;=\; \mathsf{ap}_{-\circ L(F_{i,j,g})}(Q_j) \cdot \mathsf{assoc}(h, L(r_j), L(F_{i,j,g})) \cdot \mathsf{ap}_{h\circ -}(L(K_{i,j,g}))$$

The problem is that this equality need not hold for wild categories, and we offer an example of an adjunction for which it's provably false inside HoTT (Example 14). Still, by reverse engineering the equality, we arrive at the following general property of an adjunction as a sufficient condition for it to hold.

▶ **Definition 11** ([12, ladj-is-2coher]). *The left adjoint L is* 2-coherent *if for all $h_1 : \mathsf{hom}_{\mathcal{D}}(L(X), Y)$, $h_2 : \mathsf{hom}_{\mathcal{C}}(Z, X)$, and $h_3 : \mathsf{hom}_{\mathcal{C}}(W, Z)$, the following diagram commutes:*

$$\begin{tikzcd}
(\psi(h_1) \circ h_2) \circ h_3 \arrow[r, equal, "\mathsf{assoc}(\psi(h_1), h_2, h_3)"] \arrow[d, equal, "\mathsf{ap}_{-\circ h_3}(\mathsf{nat}_{dom}(h_2, h_1))"'] & \psi(h_1) \circ (h_2 \circ h_3) \arrow[d, equal, "\mathsf{nat}_{dom}(h_2 \circ h_3, h_1)"] \\
\psi(h_1 \circ L(h_2)) \circ h_3 \arrow[d, equal, "\mathsf{nat}_{dom}(h_3, h_1 \circ L(h_2))"'] & \psi(h_1 \circ L(h_2 \circ h_3)) \arrow[d, equal, "\mathsf{ap}_{\psi}(\mathsf{ap}_{h_1\circ -}(L_{\circ}(h_2, h_3)))"] \\
\psi((h_1 \circ L(h_2)) \circ L(h_3)) \arrow[r, equal, "\mathsf{ap}_{\psi}(\mathsf{assoc}(h_1, L(h_2), L(h_3)))"'] & \psi(h_1 \circ (L(h_2) \circ L(h_3)))
\end{tikzcd} \tag{2-coh}$$

▶ Note 12. In terms of a classical biadjunction [9, Definition 9.8], Definition 11 is part of the pseudonaturality of $(\psi, \mathsf{nat}_{dom}) : \mathsf{hom}_{\mathcal{D}}(L(-), X) \Rightarrow \mathsf{hom}_{\mathcal{C}}(-, R(X))$.

Intuitvely, Definition 11 accounts for the 1-dimensional datum introduced by $L(K_{i,j,g})$. This is necessary as the definition of adjunction only accounts for 0-dimensional data. By accounting for the relevant 1-dimensional data, we can fininsh porting the standard proof to wild category theory as follows.

▶ **Theorem 13** ([12, Ladj-colim]). *Suppose that L is 2-coherent. The cocone $L(\mathcal{K})$ under $L(F)$ is colimiting.*

Proof. For all $X : \mathsf{Ob}(\mathcal{D})$, we have that

$$\begin{aligned}
& \mathsf{hom}_{\mathcal{D}}(L(C), X) \\
\simeq\; & \mathsf{hom}_{\mathcal{C}}(C, R(X)) && (\textit{hom-isomorphism}) \\
\simeq\; & \mathsf{lim}_{i:\Gamma^{\mathrm{op}}}(\mathsf{hom}_{\mathcal{C}}(F_i, R(X))) && (\mathcal{K}\ \textit{is colimiting}) \\
\simeq\; & \mathsf{lim}_{i:\Gamma^{\mathrm{op}}}(\mathsf{hom}_{\mathcal{D}}(L(F_i), X)) && (\textit{Lemma 8 applied to } \left(\psi^{-1}, \widetilde{\mathsf{nat}}_{dom}\right) \circ F)
\end{aligned}$$

Let ζ denote the composite of these three equivalences: ζ sends $h : \mathsf{hom}_{\mathcal{D}}(L(C), X)$ to

$$\left(\lambda i.\psi^{-1}(\psi(h) \circ r_i), \lambda j \lambda i \lambda g.\widetilde{\mathsf{nat}}_{dom}(F_{i,j,g}, \psi(h) \circ r_j) \cdot \mathsf{ap}_{\psi^{-1}}(\mathsf{assoc}(\psi(h), r_j, F_{i,j,g}) \cdot \mathsf{ap}_{\psi(h)\circ -}(K_{i,j,g}))\right)$$

We want to show $\zeta(h) = \mathsf{postcomp}_{L(\mathcal{K})}(X, h)$. For each $i : \Gamma_0$, we have the chain Q_i of paths

$$\psi^{-1}(\psi(h) \circ r_i) \xlongequal{\mathsf{ap}_{\psi^{-1}}(\mathsf{nat}_{dom}(r_i, h))} \psi^{-1}(\psi(h \circ L(r_i)) \xlongequal{\eta_{\psi}(h \circ L(r_i))} h \circ L(r_i)$$

By Lemma 9, it suffices to prove that

$$\begin{array}{c} \widetilde{\mathsf{nat}}_{dom}(F_{i,j,g}, \psi(h) \circ r_j) \cdot \mathsf{ap}_{\psi^{-1}}(\mathsf{assoc}(\psi(h), r_j, F_{i,j,g}) \cdot \mathsf{ap}_{\psi(h)\circ -}(K_{i,j,g})) \cdot Q_i \\ \| \\ \mathsf{ap}_{-\circ L(F_{i,j,g})}(Q_j) \cdot \mathsf{assoc}(h, L(r_j), L(F_{i,j,g})) \cdot \mathsf{ap}_{h\circ -}(L(K_{i,j,g})) \end{array} \tag{eq-edge}$$

We start with the top endpoint of (`eq-edge`). By Lemma 7, we have the commuting diagram

$$\begin{tikzcd}
\psi^{-1}(\psi(h) \circ r_j) \circ L(F_{i,j,g}) \arrow[r, equal, "\widetilde{\mathsf{nat}}_{dom}(F_{i,j,g}, \psi(h)\circ r_j)"] \arrow[d, equal, "\mathsf{ap}_{\psi^{-1}(-)\circ L(F_{i,j,g})}(\mathsf{nat}_{dom}(r_j,h))"'] \arrow[dr, phantom, "\textit{homotopy naturality}"] & \psi^{-1}((\psi(h) \circ r_j) \circ F_{i,j,g}) \arrow[d, equal, "\mathsf{ap}_{\psi^{-1}(-\circ F_{i,j,g})}(\mathsf{nat}_{dom}(r_j,h))"] \\
\psi^{-1}(\psi(h \circ L(r_j))) \circ L(F_{i,j,g}) \arrow[r, equal, "\widetilde{\mathsf{nat}}_{dom}(F_{i,j,g}, \psi(h\circ L(r_j)))"'] \arrow[d, equal, "\mathsf{ap}_{-\circ L(F_{i,j,g})}(\eta_\psi(h\circ L(r_j)))"'] \arrow[dr, phantom, "\mathsf{exch}(F_{i,j,g}, h\circ L(r_j))"] & \psi^{-1}(\psi(h \circ L(r_j)) \circ F_{i,j,g}) \arrow[d, equal, "\mathsf{ap}_{\psi^{-1}}(\mathsf{nat}_{dom}(F_{i,j,g}, h\circ L(r_j)))"] \\
(h \circ L(r_j)) \circ L(F_{i,j,g}) \arrow[r, equal, "\eta_\psi((h\circ L(r_j))\circ L(F_{i,j,g}))^{-1}"'] & \psi^{-1}(\psi((h \circ L(r_j)) \circ L(F_{i,j,g})))
\end{tikzcd}$$

We also have the commuting diagram

$$\begin{tikzcd}
\psi^{-1}(\psi(h) \circ (r_j \circ F_{i,j,g})) \arrow[r, equal, "\mathsf{ap}_{\psi^{-1}}(\mathsf{ap}_{\psi(h)\circ -}(K_{i,j,g}))"] \arrow[d, equal, "\mathsf{ap}_{\psi^{-1}}(\mathsf{nat}_{dom}(r_j\circ F_{i,j,g}, h))"'] \arrow[dr, phantom, "\textit{homotopy naturality}"] & \psi^{-1}(\psi(h) \circ r_i) \arrow[d, equal, "\mathsf{ap}_{\psi^{-1}}(\mathsf{nat}_{dom}(r_i, h))"] \\
\psi^{-1}(\psi(h \circ L(r_j \circ F_{i,j,g}))) \arrow[r, equal, "\mathsf{ap}_{\psi^{-1}}(\mathsf{ap}_{\psi}(\mathsf{ap}_{h\circ L(-)}(K_{i,j,g})))"'] \arrow[d, equal, "\eta_\psi(h\circ L(r_j\circ F_{i,j,g}))"'] \arrow[dr, phantom, "\textit{homotopy naturality}"] & \psi^{-1}(\psi(h \circ L(r_i))) \arrow[d, equal, "\eta_\psi(h\circ L(r_i))"] \\
h \circ L(r_j \circ F_{i,j,g}) \arrow[r, equal, "\mathsf{ap}_{h\circ -}(\mathsf{ap}_L(K_{i,j,g}))"'] & h \circ L(r_i)
\end{tikzcd}$$

By rewriting (`eq-edge`) with these two commuting diagrams, we turn it into the equality expressing that the following diagram commutes:

$$\begin{tikzcd}
\psi^{-1}((\psi(h) \circ r_j) \circ F_{i,j,g}) \arrow[r, equal, "\mathsf{ap}_{\psi^{-1}}(\mathsf{assoc}(\psi(h), r_j, F_{i,j,g}))"] \arrow[d, equal, "\mathsf{ap}_{\psi^{-1}(-\circ F_{i,j,g})}(\mathsf{nat}_{dom}(r_j,h))"'] & \psi^{-1}(\psi(h) \circ (r_j \circ F_{i,j,g})) \arrow[d, equal, "\mathsf{ap}_{\psi^{-1}}(\mathsf{nat}_{dom}(r_j\circ F_{i,j,g}, h))"] \\
\psi^{-1}(\psi(h \circ L(r_j)) \circ F_{i,j,g}) \arrow[d, equal, "\mathsf{ap}_{\psi^{-1}}(\mathsf{nat}_{dom}(F_{i,j,g}, h\circ L(r_j)))"'] & \psi^{-1}(\psi(h \circ L(r_j \circ F_{i,j,g}))) \arrow[d, equal, "\eta_\psi(h\circ L(r_j\circ F_{i,j,g}))"] \\
\psi^{-1}(\psi((h \circ L(r_j)) \circ L(F_{i,j,g}))) \arrow[d, equal, "\eta_\psi((h\circ L(r_j))\circ L(F_{i,j,g}))"'] & h \circ L(r_j \circ F_{i,j,g}) \arrow[d, equal, "\mathsf{ap}_{h\circ -}(L_\circ(r_j, F_{i,j,g}))"] \\
(h \circ L(r_j)) \circ L(F_{i,j,g}) \arrow[r, equal, "\mathsf{assoc}(h, L(r_j), L(F_{i,j,g}))"'] & h \circ (L(r_j) \circ L(F_{i,j,g}))
\end{tikzcd} \tag{†}$$

At this point, we could have defined (`2-coh`) so that it aligns with (†), but this equality is difficult to check in practice. Hence we now transform (†) into something more tractable. Since ψ is an equivalence (hence an embedding), the equality (†) is equivalent to its image under ap_ψ. By homotopy naturality of η_ψ, this image is equivalent to an equality expressing that the following diagram commutes:

$$\begin{tikzcd}
(\psi(h) \circ r_j) \circ F_{i,j,g} \arrow[r, equal, "\mathsf{assoc}(\psi(h), r_j, F_{i,j,g})"] \arrow[d, equal, "\mathsf{ap}_{-\circ F_{i,j,g}}(\mathsf{nat}_{dom}(r_j, h))"'] & \psi(h) \circ (r_j \circ F_{i,j,g}) \arrow[d, equal, "\mathsf{nat}_{dom}(r_j \circ F_{i,j,g}, h)"] \\
\psi(h \circ L(r_j)) \circ F_{i,j,g} \arrow[d, equal, "\mathsf{nat}_{dom}(F_{i,j,g}, h \circ L(r_j))"'] & \psi(h \circ L(r_j \circ F_{i,j,g})) \arrow[d, equal, "\mathsf{ap}_{\psi}(\mathsf{ap}_{h\circ -}(L_{\circ}(r_j, F_{i,j,g})))"] \\
\psi((h \circ L(r_j)) \circ L(F_{i,j,g})) \arrow[r, equal, "\mathsf{ap}_{\psi}(\mathsf{assoc}(h, L(r_j), L(F_{i,j,g})))"'] & \psi(h \circ (L(r_j) \circ L(F_{i,j,g})))
\end{tikzcd}$$

Finally, this diagram commutes because L is 2-coherent. ◀

▶ **Example 14.** As claimed, without the 2-coherence assumption, (`eq-edge`) may not hold.

Define the wild category $\mathcal{E}$ by $\mathsf{Ob}(\mathcal{E}) :\equiv \mathbf{1}$ and $\hom_{\mathcal{E}}(*, *) :\equiv S^1$. Here, S^1 denotes the circle, defined as the HIT generated by a point $\mathsf{base} : S^1$ and a path $\mathsf{loop} : \mathsf{base} = \mathsf{base}$. The remaining structure on $\mathcal{E}$, including its composition operation $\bullet$, comes from path concatenation on a loop space. Indeed, S^1 is the loop space of the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ [14]. For all $\ell : \hom_{\mathcal{E}}(*, *)$, we have a nontrivial loop loop_{ℓ} at ℓ: It is known that loop is nontrivial [24, Lemma 6.4.1], and we have an equivalence $f : (S^1, \mathsf{base}) \to (S^1, \ell)$ of pointed types because S^1 is a loop space. (As we'll see in Section 6, loop spaces are independent of basepoint in this sense, i.e., *homogeneous*.)

Let the functor $\Lambda : \mathcal{E} \to \mathcal{E}$ be the identity on objects and morphisms, but let $\Lambda_{\circ}(\ell_1, \ell_2) :\equiv \mathsf{loop}_{\ell_1 \bullet \ell_2}$. Consider the evident adjunction $\Lambda \dashv \Lambda$. If $h \equiv \mathsf{id}_*$, then (`eq-edge`), with respect to this adjunction, reduces to showing $\Lambda_{\circ}(h_2, h_3)$ is trivial. But it's nontrivial by construction.

6 Suspension is 2-coherent

This section (along with Section 7) offers evidence that Definition 11 is useful in practice. We show that the suspension endofunctor $\Sigma : \mathcal{U}_* \to \mathcal{U}_*$ on the wild category of pointed types is a 2-coherent left adjoint to the loop space endofunctor Ω (which maps (X, x_0) to $(x_0 = x_0, \mathsf{refl}_{x_0})$). By Theorem 13, we deduce that Σ preserves (graph-indexed) colimits. Although the diagram (`2-coh`) is a valuable approach to proving this preservation property, it does rely on a new trick based on *homogeneous types* to handle the path algebra generated by Σ. The adjunction $\Sigma \dashv \Omega$ is already known [8, Lemma 2.16], so our contribution is verifying that Σ is 2-coherent. Also, to our knowledge, ours is the first proof in HoTT (in particular, Book HoTT) that Σ preserves colimits.

If X is a pointed type, then $\mathsf{ty}(X)$ and $\mathsf{pt}(X)$ denote the underlying type and basepoint of X, respectively. If $f : X \to_* Y$ is a pointed map, then $\mathsf{fun}(f)$ and $\mathsf{bp}(f)$ denote the underlying function and proof of basepoint preservation of f, respectively.

▶ **Definition 15.** *Let $f_1, f_2 : X_1 \to_* X_2$ be pointed maps. A* pointed homotopy $f_1 \sim_* f_2$ *is a homotopy $H : \mathsf{fun}(f_1) \sim \mathsf{fun}(f_2)$ with a path $H(\mathsf{pt}(X_1)) \cdot \mathsf{bp}(f_2) = \mathsf{bp}(f_1)$.*

The SIP for pointed maps says that the canonical function $f_1 = f_2 \to f_1 \sim_* f_2$ is an equivalence, with inverse denoted by $\langle -, - \rangle$. We use this equivalence to define the terms $\Sigma_{\circ}$ and nat_{dom}, which will help us manipulate them in the proof of 2-coherence.

The composition law's first component is defined by induction (see Section 3.2), with t_{N}

and t_S trivial and T defined, for suitable pointed maps r and s, via the commuting square

$$\begin{tikzcd}
\mathsf{ap}_{\Sigma(s\circ r)}(\mathsf{glue}(x)) \arrow[r, equal] \arrow[d, equal, "\rho_{\Sigma(s\circ r)}(x)"'] & \mathsf{ap}_{\Sigma(s)\circ\Sigma(r)}(\mathsf{glue}(x)) \arrow[d, equal, "via\ \rho_{\Sigma(r)}(x)"] \\
\mathsf{glue}(\mathsf{fun}(s)(\mathsf{fun}(r)(x))) \arrow[r, equal, "via\ \rho_{\Sigma(s)}(\mathsf{fun}(r)(x))"'] & \mathsf{ap}_{\Sigma(s)}(\mathsf{glue}(\mathsf{fun}(r)(x)))
\end{tikzcd}$$

Its second component is trivial. Next, the adjunction $\Sigma \dashv \Omega$ [12, SuspAdjointLoop] is defined component-wise by

$$\begin{aligned}
\Phi \ &: \ \mathsf{hom}_{\mathcal{U}_*}(\Sigma(X,x_0),(Y,y_0)) \xrightarrow{\simeq} \mathsf{hom}_{\mathcal{U}_*}((X,x_0),\Omega(Y,y_0)) \\
\Phi(h,h_0) \ &:= \ (\lambda x. \underbrace{h_0^{-1}\cdot \mathsf{ap}_h(\mathsf{glue}(x)\cdot \mathsf{glue}(x_0)^{-1})\cdot h_0}_{\xi(x,h,h_0)}, \zeta(\mathsf{glue}(x_0),h_0))
\end{aligned}$$

where the underbrace denotes term abbreviation and $\zeta(\mathsf{glue}(x_0), h_0)$ denotes the evident path of type $\xi(x_0, h, h_0) = \mathsf{refl}_{y_0}$. For all $f^* :\equiv (f, f_0) : (Z, z_0) \to_* (X, x_0)$ and $h^* :\equiv (h, h_0) : \Sigma(X, x_0) \to_* (Y, y_0)$, $\mathsf{nat}_{dom}(f^*, h^*)$ is defined as the path

$$\begin{aligned}
&\Phi(h^*)\circ f^* \\
\equiv\ &\Big(\lambda x.\xi(f(x),h^*), \mathsf{ap}_{\xi(-,h^*)}(f_0)\cdot\zeta(\mathsf{glue}(x_0),h_0)\Big) \\
=\ &(\lambda x.\xi(x, h\circ \mathsf{fun}(\Sigma(f^*)),h_0), \zeta(\mathsf{glue}(z_0),h_0)) && (\langle\Theta,\Theta_0\rangle) \\
\equiv\ &\Phi(h\circ\mathsf{fun}(\Sigma(f^*)),h_0) \\
\equiv\ &\Phi(h^*\circ\Sigma(f^*))
\end{aligned}$$

Here, for each $z : Z$, $\Theta(z)$ is defined as the path

$$\begin{aligned}
&h_0^{-1}\cdot\mathsf{ap}_h(\mathsf{glue}(f(z))\cdot\mathsf{glue}(f(z_0))^{-1})\cdot h_0 \\
&\qquad\qquad \big\|\ via\ \rho_{\Sigma(f^*)}(z)\ and\ \rho_{\Sigma(f^*)}(z_0) \\
&h_0^{-1}\cdot\mathsf{ap}_{h\circ\mathsf{fun}(\Sigma(f^*))}(\mathsf{glue}(z)\cdot\mathsf{glue}(z_0)^{-1})\cdot h_0
\end{aligned}$$

and Θ_0 is defined by path induction on $\rho_{\Sigma(f^*)}(z_0)$.

Now that we've defined $\Sigma_\circ$ and nat_{dom}, we claim that the diagram (2-coh) commutes. The SIP for pointed homotopies turns this goal into a *double pointed homotopy*:

▶ **Definition 16.** *Let f_1 and f_2 be pointed maps and let $(H_1,\kappa_1),(H_2,\kappa_2) : f_1 \sim_* f_2$. A* double pointed homotopy $(H_1,\kappa_1) \sim^2_* (H_2,\kappa_2)$ *consists of a homotopy $\mu : H_1 \sim H_2$ and a commuting triangle*

$$\begin{tikzcd}
H_2(\mathsf{pt}(X_1))\cdot\mathsf{bp}(f_2) \arrow[d, equal, "\mathsf{ap}_{-\cdot\mathsf{bp}(f_2)}(\mu(\mathsf{pt}(X_1)))"'] \arrow[dr, equal, "\kappa_2"] & \\
H_1(\mathsf{pt}(X_1))\cdot\mathsf{bp}(f_2) \arrow[r, equal, "\kappa_1"'] & \mathsf{bp}(f_1)
\end{tikzcd}$$

To construct the first component of the desired double pointed homotopy, we have to reduce a large expression involving various ρ terms (coming from the nat_{dom} and $\Sigma_\circ$ edges of (2-coh)). We do so via a mechanical, though nontrivial, process of iteratively eliminating matching ρ terms. The commuting triangle, however, is infeasible to construct directly. The problem is that it contains the entire first component, which involves complex path algebra and does not reduce at $\mathsf{pt}(W)$. In addition, it contains a handful of nontrivial path inductions from Θ's second component. The result is a term that is simply too big.

Luckily, we can get the commuting triangle for free by noticing the special nature of loop spaces. Every loop space is a *homogeneous type*, i.e., a pointed type (X, x_0) equipped with a pointed equivalence $\mathsf{auto}_x : (X, x_0) \xrightarrow{\simeq}_* (X, x)$ for every $x : X$. In this case, we also say X is *homogeneous at* x_0. We note a few things about such types. First, if M is homogeneous, then it's homogeneous at all its elements. Second, by applying $- \circ \mathsf{auto}^{-1}_{\mathsf{pt}(M)}$ to auto, we can make every homogeneous type M *strongly homogeneous*, i.e., $\mathsf{auto}_{\mathsf{pt}(M)} = \mathsf{id}_M$. Finally, as Ω preserves pointed equivalences, if (X, x_0) is homogeneous, we have $\mathsf{homog\text{-}pth}(x_0, x) : (x_0 = x_0) \xrightarrow{\simeq} (x = x)$ for every $x : X$. Now, a key insight for our goal is Cavallo's trick: two pointed maps into homogoneous types are pointed-homotopic when their underlying functions are homotopic [23, $\rightarrow$·Homogeneous≡]. As our goal is a higher pointed homotopy, we want the following higher version of the trick, which will finish the proof that Σ is 2-coherent [12, Susp-2coher]:

▶ **Lemma 17** ([12, $\sim\odot$homog$\sim$]). *Let $f_1, f_2 : X_1 \to_* X_2$ with X_2 homogeneous. Let $(H_1, \kappa_1), (H_2, \kappa_2) : f_1 \sim_* f_2$. If $H_1 \sim H_2$, then $(H_1, \kappa_1) \sim^2_* (H_2, \kappa_2)$.*

Proof. We begin with a general observation. Let $k : X_1 \to_* X_2$ and consider the evaluation map $\mathsf{ev}_{\mathsf{pt}(X_1),\mathsf{fun}(k)} : (\mathsf{fun}(k) \sim \mathsf{fun}(k), \mathsf{refl}) \to_* \Omega(X_2, \mathsf{fun}(k)(\mathsf{pt}(X_1)))$. As X_2 is homogeneous at $\mathsf{fun}(f_1)(\mathsf{pt}(X_1))$, this map has a pointed section σ^* whose underlying function sends a loop p at $\mathsf{fun}(k)(\mathsf{pt}(X_1))$ to the homotopy $\sigma(p, x) :\equiv \mathsf{homog\text{-}pth}(\mathsf{fun}(k)(\mathsf{pt}(X_1)), \mathsf{fun}(k)(x), p)$. It's easy to check that σ is pointed. It remains to construct a pointed homotopy $\gamma : \mathsf{ev}_{\mathsf{pt}(X_1),\mathsf{fun}(k)} \circ \sigma^* \sim_* \mathsf{id}$. The first component of γ is a homotopy $\mathsf{homog\text{-}pth}(\mathsf{pt}(X_1), \mathsf{pt}(X_1)) \sim \mathsf{id}_{\mathsf{fun}(k)(\mathsf{pt}(X_1))=\mathsf{fun}(k)(\mathsf{pt}(X_1))}$, which we get by promoting X_2 to a strongly homogeneous type. The second component, which also uses the fact X_2 is strongly homogeneous, follows routinely.

Let $Q : H_1 \sim H_2$. By strong function extensionality, we "path induct" on H_1 and Q so that they are both identity homotopies. Further, by generalizing $\mathsf{pt}(X_2)$, we induct on $\mathsf{bp}(f_1)$ to make it refl_{w_0} with $w_0 :\equiv \mathsf{fun}(f_1)(\mathsf{pt}(X_1))$. By our general observation, $\mathsf{ev}_{\mathsf{pt}(X_1),\mathsf{fun}(f_1)}$ has a pointed section σ^*, so that $\Omega(\sigma^*)$ is a pointed section of $\Omega(\mathsf{ev}_{\mathsf{pt}(X_1),\mathsf{fun}(f_1)})$. We want a pair $(\mu, \mu_0) : (\mathsf{refl}, \kappa_1) \sim^2_* (\mathsf{refl}, \kappa_2)$. We define $\mu : \mathsf{refl} \sim \mathsf{refl}$ as the image $\mathsf{fun}(\Omega(\sigma^*))(\kappa)$ of a certain loop κ at refl_{w_0} under $\Omega(\sigma^*)$. To make the right choice for κ, we look ahead to the commuting triangle μ_0:

$$\begin{tikzcd}
\mathsf{bp}(f_1) \arrow[d, equal, "\mathsf{ap}_{-\cdot\mathsf{bp}(f_1)}(\mu(\mathsf{pt}(X_1)))"'] \arrow[dr, equal, "\kappa_2"] & \\
\mathsf{bp}(f_1) \arrow[r, equal, "\kappa_1"'] & \mathsf{refl}_{w_0}
\end{tikzcd}$$

Since $\Omega(\sigma^*)$ is a pointed section of $\Omega(\mathsf{ev}_{\mathsf{pt}(X_1),\mathsf{fun}(f_1)})$, $\mu(\mathsf{pt}(X_1))$ will equal κ. Finally, $-\cdot\mathsf{bp}(f_1)$ is an equivalence, so we simply solve for κ. ◀

▶ **Theorem 18** ([12, Susp-colim]). *The suspension $\Sigma : \mathcal{U}_* \to \mathcal{U}_*$ preserves colimits.*

Recall that a type is *acyclic* if its suspension is contractible [3]. We contribute a new closure property of acyclic types with the next corollary. It uses Hart and Favonia's construction of colimits colim^* in $\mathcal{U}_*$ as the cofiber of a map between colimits colim in $\mathcal{U}$ [11, Theorem 15]. (Colimits in $\mathcal{U}$ are postulated as HITs definable from pushouts.)

▶ **Corollary 19.** *The class of pointed acyclic types is closed under colimits in $\mathcal{U}_*$.*

Proof. By Theorem 18 and uniqueness of colimits in $\mathcal{U}_*$ (as in any wild bicategory), we have an equivalence of pointed types $\Sigma(\mathsf{colim}^*(F)) \simeq_* \mathsf{colim}^*(\Sigma(F))$. If $\mathsf{ty}(F_i)$ is acyclic for each $i : \Gamma_0$, then $\mathsf{ty}(\mathsf{colim}^*(\Sigma(F)))$ is contractible as the cofiber of an equivalence, namely the function $\mathsf{colim}(\mathbf{1}) \to \mathsf{colim}(\mathsf{ty} \circ \Sigma(F))$ induced by the unique natural transformation into $\mathsf{ty} \circ \Sigma(F)$. This completes the proof since equivalences preserve contractibility. ◀

▶ Note 20. Theorem 18 also puts on firm footing a key step of Graham's construction of stable homotopy as a homology theory: proving that Σ preserves cofibers [10, Corollary 2.2]. Graham presents an ad-hoc pen-and-paper proof that omits some complex equality proofs. By contrast, we offer a fully mechanized proof based on a general property of left adjoints.

7 Colimits of modal types

We end with another application of Theorem 13 in synthetic homotopy theory. We prove that all *modalities* on coslices of a universe $\mathcal{U}$ are 2-coherent and thereby construct (graph-indexed) colimits of modal types. Consider functions $\bigcirc : \mathcal{U} \to \mathcal{U}$ and $\eta : \prod_{X:\mathcal{U}} X \to \bigcirc X$. A type $X : \mathcal{U}$ is *modal* if η_X is an equivalence. Let $\mathcal{U}_{\bigcirc}$ denote the subuniverse of modal types.

▶ **Definition 21** ([12, Modality]). *We say that $\bigcirc$ is a* modality *if*

- *for all $X : \mathcal{U}$, $\bigcirc X$ is modal;*
- *for all $X : \mathcal{U}$ and $x, y : \bigcirc X$, the identity type $x = y$ is modal;*
- *for all $X : \mathcal{U}$ and $P : \bigcirc X \to \mathcal{U}_{\bigcirc}$, the function $- \circ \eta_X : \left(\prod_{x:\bigcirc X} P(x)\right) \to \left(\prod_{x:X} P(\eta_X(x))\right)$ has a section. (This condition is called* $\bigcirc$-induction.*)*

Suppose $\bigcirc$ is a modality and let A be a type. By $\bigcirc$-induction (including the associated nondependent recursion principle), we have a functor $\bigcirc^A : A/\mathcal{U} \to (A/\mathcal{U})_{\bigcirc}$ into the full wild subcategory of $A/\mathcal{U}$ on those (X, s) with X modal, called *modal A-types*. It is a straightforward extension of the functor $\bigcirc : \mathcal{U} \to \mathcal{U}_{\bigcirc}$: for example, its object function sends (X, s) to $(\bigcirc X, \eta_X \circ s)$. By $\bigcirc$-induction, we also have a family of equivalences $\left(\bigcirc^A U \to_A V\right) \xrightarrow{\simeq} (U \to_A V)$ for A-types U and modal A-types V that is natural in U. (It's trivially natural in V.) Hence we have an adjunction between $\bigcirc^A$ and the forgetful functor [12, Mod-cos-adj].

▶ **Theorem 22** ([12, Mod-cos-adj-2coh]). *The left adjoint $\bigcirc^A$ is 2-coherent.*

Proof. By $\bigcirc$-induction followed by a burst of path induction. ◀

▶ **Corollary 23** ([12, Mod-colim]). *The wild category $(A/\mathcal{U})_{\bigcirc}$ has all colimits.*

Proof. Let F be a Γ-shaped diagram in $(A/\mathcal{U})_{\bigcirc}$. By [11, Theorem 15], the diagram $\mathcal{F}_{\bigcirc,A}(F)$ has a colimit $\mathsf{colim}^A(F)$ in $A/\mathcal{U}$. Since $\bigcirc^A$ preserves colimits (Theorem 22), we have the following colimiting cocone and natural isomorphism in $(A/\mathcal{U})_{\bigcirc}$:

$$\begin{tikzcd}
\bigcirc^A(\mathcal{F}(F_i)) \arrow[rr, "\bigcirc^A(\mathcal{F}(F_{i,j,g}))"] \arrow[dr] & & \bigcirc^A(\mathcal{F}(F_i)) \arrow[dl] \\
& \bigcirc^A(\mathsf{colim}^A(F)) &
\end{tikzcd}
\qquad
\begin{tikzcd}
F_i \arrow[r, "F_{i,j,g}"] \arrow[d, "\eta^A", "\simeq"'] & F_j \arrow[d, "\eta^A"', "\simeq"] \\
\bigcirc^A(\mathcal{F}(F_i)) \arrow[r, "\bigcirc^A(\mathcal{F}(F_{i,j,g}))"'] & \bigcirc^A(\mathcal{F}(F_j))
\end{tikzcd}$$

where $\eta^A(Z) :\equiv \left(\eta_{\mathsf{ty}(Z)}, \mathsf{refl}\right) : Z \to_A \bigcirc^A(Z)$ for all $Z : A/\mathcal{U}$. In univalent wild bicategories, composing with natural isomorphisms preserves colimiting cocones. Indeed, univalence lets us reduce the natural isomorphism $F \Rightarrow \bigcirc^A \circ \mathcal{F}$ to the identity, and a standard property of bicategories implies that composing with the identity preserves colimiting cocones. Finally, we note $(A/\mathcal{U})_{\bigcirc}$ is a wild bicategory and, assuming the univalence axiom, a univalent one. ◀

Our construction of colimits of modal types is simpler than the Book proof (see Section 1.2.3) and illuminates a higher coherence used by the latter. Equality (7.4.11) of the Book proof secretly requires a coherence condition between $\|-\|_n$'s composition law and its naturality data nat_n, which is satisfied because $\|g \circ f\|_n \circ |-|_n \equiv |-|_n \circ g \circ f$. The required coherence has a similar flavor to Definition 11, but our proof makes such a condition explicit as part of a general framework.

8 Conclusion and future work

We addressed a coherence problem in the proof that a left adjoint between wild categories preserves colimits. We proved that the coherence, which always holds in the classical setting, may be false for wild categories. With just "off-the-shelf" tools from HoTT, we identified a relatively tractable sufficient condition on the left adjoint for the proof to work, namely 2-*coherence*. We showed that the suspension functor is 2-coherent and thus preserves colimits. In doing so, we managed to avoid an infeasible equality proof by developing a higher-dimensional version of Cavallo's trick for homogeneous types. Finally, we showed that modalities on coslices of a universe are 2-coherent and, as a result, that the associated subcategories of modal types are cocomplete.

There are a few open questions raised by our work. The simplest is the analysis of the dual statement that right adjoints preserve limits for wild categories, which should be similar to the one presented here. Another question is whether we can extend Theorem 22 to all reflective subuniverses. Finally, it would be quite useful to find a trick to show, in Book HoTT, that the smash product $\bullet \wedge - : \mathcal{U}_* \to \mathcal{U}_*$ is 2-coherent. The right adjoint of the smash product is the pointed map space functor, which is not generally valued in homogeneous types. Hence we cannot use Lemma 17 to escape the infeasible equality proof, which is likely even harder than the one for the suspension.

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