Coslice Colimits in Homotopy Type Theory

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Goals

1. Construct coslice colimits in a way that reveals their relation to colimits in a type universe $\mathcal{U}$.
2. Use the construction to prove elegant categorical results about colimits *(not in this talk)*.
3. Use the construction to prove useful results about other areas of synthetic homotopy theory:
   - **factorization systems**
   - **higher group theory**
   - **cohomology theory** *(not in this talk)*

Background

- **Homotopy theory** deals with topological spaces and maps up to continuous deformation.
- **Homotopy type theory (HoTT)** is a formal system
  - for **reasoning synthetically** about homotopy theory, i.e., for *synthetic homotopy theory*
  - equipped with **semi-decidable proof-checking**, with implementations in Agda and Coq.

Our work:

Use HoTT to build verified proofs of theorems in homotopy theory.

Focus of our work: coslices

Let $\mathcal{C}$ be a category and A an object.

The *coslice* $A/\mathcal{C}$ *of* $\mathcal{C}$ *under* A has objects and morphisms

objects: $$\begin{array}{c} A \\ {\scriptstyle f_B}\big\downarrow \\ B \end{array}$$ **morphisms:** $$\begin{array}{ccc} & A & \\ {\scriptstyle f_{B_1}}\swarrow & & \searrow{\scriptstyle f_{B_2}} \\ B_1 & \xrightarrow[\varphi]{} & B_2 \end{array}$$

Coslices of the category of spaces appear often in homotopy theory.

In particular, the category of pointed spaces, i.e., the coslice under the one-point space.

In HoTT, a type universe $\mathcal{U}$ forms a ***(wild) category***.

Coslices of $\mathcal{U}$ have natural roles in synthetic homotopy theory.

So do their ***colimits***.

Fundamental constructions from category theory, letting us build complex spaces from simpler ones.

This work:

theory of **coslice colimits** inside HoTT,[1] i.e., colimits in $A/\mathcal{U}$

[1] Due to open questions about the definability of general ∞-categories in HoTT, all colimits are over free categories in this work.

Formal setting

HoTT extends Martin-Löf type theory (MLTT) with

- **the univalence axiom** and/or
- **higher inductive types (HITs)**, e.g., the *colimit type*.
  Generalize inductive types by allowing constructors of Id types.

Our coslice colimit construction is done in a small variant of HoTT:

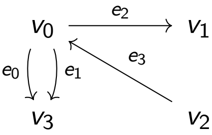
$$\mathsf{MLTT} + \mathsf{Colimit}$$

Technical note: Pushouts are enough to define all colimits as HITs.

Colimit type

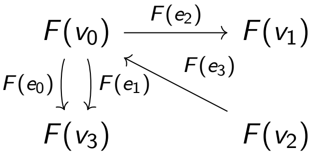
A ***(directed) graph*** is a pair $\Gamma :\equiv (\Gamma_0, \Gamma_1)$ consisting of

- a type $\Gamma_0 : \mathcal{U}$ of vertices
- a family $\Gamma_1 : \Gamma_0 \to \Gamma_0 \to \mathcal{U}$ of edges.



$v_i \, : \, \Gamma_0$

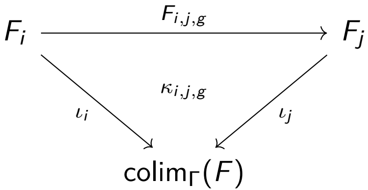
Let F be a Γ-shaped diagram in $\mathcal{U}$.



$F(v_i) \, : \, \mathcal{U}$

The ***colimit of*** F is the HIT $\mathrm{colim}_\Gamma(F)$ generated by

$$
\begin{aligned}
\iota &: (i : \Gamma_0) \to F_i \to \mathrm{colim}_\Gamma(F) \\
\kappa &: (i, j : \Gamma_0)\,(g : \Gamma_1(i,j)) \to \iota_j \circ F_{i,j,g} \sim \iota_i
\end{aligned}
$$



This data forms a ***cocone on*** $\mathrm{colim}_\Gamma(F)$ ***under*** F.

Coslices and coslice colimits

Let $A : \mathcal{U}$. We have a ***coslice category*** $A/\mathcal{U}$ defined by

$$\mathrm{Ob}(A/\mathcal{U}) := \sum_{X:\mathcal{U}} A \to X$$

$$\underbrace{\hom_{A/\mathcal{U}}(Y_1, Y_2)}_{Y_1 \to_A Y_2} := \sum_{f:\mathrm{pr}_1(Y_1)\to\mathrm{pr}_1(Y_2)} f \circ \mathrm{pr}_2(Y_1) \sim \mathrm{pr}_2(Y_2)$$

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Let Γ be a graph and F be a Γ-shaped diagram in $A/\mathcal{U}$.

An F-cocone K on an object Y of $A/\mathcal{U}$ is a ***colimit of*** F if for each $X : \mathrm{Ob}(A/\mathcal{U})$, the evident function

$$\mathrm{postcomp}(K, X) \ : \ (Y \to_A X) \ \to \ \mathrm{Cocone}_F(X)$$

is an equivalence of types.

What does an F-cocone (h, H) look like in $A/\mathcal{U}$?

The commuting triangle $H_{i,j,g} : h_j \circ F_{i,j,g} \sim_A h_i$ consists of

- a homotopy (pointwise equality)

$$\eta_{i,j,g} : \mathsf{pr}_1(h_j) \circ \mathsf{pr}_1(F_{i,j,g}) \sim \mathsf{pr}_1(h_i)$$

  of the underlying functions

- for each $a : A$, a 2-cell (equality between equalities) involving $\eta_{i,j,g}(\mathsf{pr}_2(F_i)(a))$ and the data of F and h.

  Distinguishes the colimit of F, in $A/\mathcal{U}$, from $\mathrm{colim}_\Gamma(\mathcal{S}(F))$

  $\mathcal{S} : A/\mathcal{U} \to A$ is the *forgetful* functor.

Question: How should we construct the colimit of F in $A/\mathcal{U}$?

1. Directly define it as a 2-dimensional HIT.

   Sure, but unhelpful.

2. Apply the forgetful functor $\mathcal{S} : A/\mathcal{U} \to \mathcal{U}$ to F and then form the colimit in $\mathcal{U}$ of $\mathcal{S}(F)$ augmented by $\mathrm{pr}_1(A) \to \bullet$.

   This well-known construction won't work.

3. Form the colimit $\mathrm{colim}_\Gamma(\mathcal{S}(F))$ and then attach 2-cells to it.

   Works nicely. ***We take this approach****.*

Correct approach

1. Form the pushout square

$$\begin{array}{ccc} \mathrm{colim}_\Gamma A & \xrightarrow{\psi} & \mathrm{colim}_\Gamma(\mathcal{S}(F)) \\ {\scriptstyle \langle \mathrm{id}_A \rangle_{i:\Gamma_0}} \downarrow & \mathrm{glue}_{\mathcal{P}_F} & \downarrow {\scriptstyle \mathrm{inr}} \\ A & \xrightarrow[\mathrm{inl}]{} & \mathcal{P}_F \end{array}$$

2. Form an F-cocone structure $\mathcal{K}(\mathcal{P}_F)$ on $(\mathcal{P}_F, \mathrm{inl})$

$$\begin{array}{ccc} F_i & \xrightarrow{F_{i,j,g}} & F_j \\ {\scriptstyle (\mathrm{inr} \circ \iota_i, \tau_i)} \searrow & & \swarrow {\scriptstyle (\mathrm{inr} \circ \iota_j, \tau_j)} \\ & \mathcal{P}_F & \end{array}$$

$$(\tau_i(a) :\equiv \mathrm{glue}_{\mathcal{P}_F}(\iota_i(a))^{-1})$$

via the cocone data on $\mathrm{colim}_\Gamma(\mathcal{S}(F))$ and the computation rules for colimit induction.

Theorem (Agda formalized)

The function

$$\mathsf{postcomp}(\mathcal{K}(\mathcal{P}_F), T) \;:\; ((\mathcal{P}_F, \mathsf{inl}) \to_A T) \;\to\; \mathsf{Cocone}_F(T)$$

is an equivalence for every $T : \mathsf{Ob}(A/\mathcal{U})$*.*

Proof.

By direct construction of a two-sided inverse. □

Action on maps

So far, we have defined a function

$$\mathrm{colim}_{\Gamma}^{A} := \mathcal{P} \ : \ \mathrm{Ob}(\mathrm{Diag}(\Gamma, A/\mathcal{U})) \to \mathrm{Ob}(A/\mathcal{U})$$

Next, make $\mathcal{P}$ a functor by defining its action on maps of diagrams.

Goal: Describe this action in terms of the action of the $\mathcal{U}$-valued colimit functor.

Let F and G be Γ-shaped diagrams in $A/\mathcal{U}$.

Let $\delta : F \Rightarrow_A G$ be a morphism of diagrams.

1. Colimit induction yields map of spans

$$\begin{array}{ccccc} A & \longleftarrow & \mathrm{colim}_\Gamma A & \longrightarrow & \mathrm{colim}_\Gamma(\mathcal{S}(F)) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \bar{\delta}} \\ A & \longleftarrow & \mathrm{colim}_\Gamma A & \longrightarrow & \mathrm{colim}_\Gamma(\mathcal{S}(G)) \end{array}$$

$\bar{\delta}$:= **action of $\mathcal{U}$-valued colimit on δ**

2. Universal property of pushouts yields map in $A/\mathcal{U}$

$$\mathrm{colim}^A_\Gamma(\delta) \ : \ \mathcal{P}_F \to_A \mathcal{P}_G$$

Make sure that this action is correct: Prove that

$$\mathrm{colim}_\Gamma^A \dashv \mathrm{const}_\Gamma$$

Amounts to two naturality squares.

Hard square:

Lemma (*Agda formalized*)

For every $T : \mathrm{Ob}(A/\mathcal{U})$ *and* $\delta : F \Rightarrow_A G$*, the following commutes:*

$$\begin{array}{ccc}
\mathrm{colim}_\Gamma^A(G) \to_A T & \xrightarrow{-\circ \mathrm{colim}_\Gamma^A(\delta)} & \mathrm{colim}_\Gamma^A(F) \to_A T \\
{\scriptstyle \mathrm{postcomp}(K(\mathcal{P}_G),T)} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{postcomp}(K(\mathcal{P}_F),T)} \\
\mathrm{Cocone}_G(T) & \xrightarrow[\mathrm{Cocone}^T(-\circ\delta)]{} & \mathrm{Cocone}_F(T)
\end{array}$$

Interaction with (orthogonal) factorization systems

Let $(\mathcal{L}, \mathcal{R})$ be an OFS on $\mathcal{U}$.

(Every map in $\mathcal{U}$ can be uniquely factored as a function in $\mathcal{L}$ followed by one in $\mathcal{R}$.)

Consider diagrams $F, G : \mathsf{Diag}(\Gamma, \mathcal{U})$.

Define the predicates on natural transformations $F \Rightarrow G$

$$\begin{aligned} \widehat{\mathcal{L}}(H, \gamma) &:= (i : \Gamma_0) \to \mathcal{L}(H_i) \\ \widehat{\mathcal{R}}(H, \gamma) &:= (i : \Gamma_0) \to \mathcal{R}(H_i) \end{aligned}$$

Lemma

The pair $\left(\widehat{\mathcal{L}}, \widehat{\mathcal{R}}\right)$ *forms an OFS on* $\mathsf{Diag}(\Gamma, \mathcal{U})$.

Corollary

The colimit functor colim_Γ *takes* $\widehat{\mathcal{L}}$ *to* $\mathcal{L}$.

For all $X, Y : \mathrm{Ob}(A/\mathcal{U})$, define the predicate

$$\mathcal{L}_A(f, p) :\equiv \mathcal{L}(f)$$

on $X \to_A Y$. Define $\widehat{\mathcal{L}}_A$ levelwise as before.

The functor colim^A_Γ **takes** $\widehat{\mathcal{L}}_A$ **to** $\mathcal{L}_A$**.**

Indeed, consider a map $\delta : F \Rightarrow_A G$ of $A/\mathcal{U}$-valued diagrams.

1. The underlying function of $\mathrm{colim}^A_\Gamma(\delta)$ is induced by

$$\begin{array}{ccccc}
A & \longleftarrow & \mathrm{colim}_\Gamma A & \longrightarrow & \mathrm{colim}_\Gamma(\mathcal{S}(F)) \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \bar{\delta}} \\
A & \longleftarrow & \mathrm{colim}_\Gamma A & \longrightarrow & \mathrm{colim}_\Gamma(\mathcal{S}(G))
\end{array}$$

2. If δ is in $\widehat{\mathcal{L}}_A$, all three vertical functions are in $\mathcal{L}$.
3. Since a map of spans is a map of diagrams, $\mathrm{colim}^A_\Gamma(\delta)$ is in $\mathcal{L}_A$.

A type $X : \mathcal{U}$ is $(\mathcal{L}, \mathcal{R})$-***connected*** if $X \to \mathbf{1}$ is in $\mathcal{L}$.

Let F be a Γ-shaped diagram of pointed $(\mathcal{L}, \mathcal{R})$-connected types.

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Let F be a Γ-shaped diagram of pointed $(\mathcal{L}, \mathcal{R})$-connected types.

The type $\mathrm{colim}^{\mathbf{1}}_{\Gamma}(\mathbf{1})$ is trivial:

$$\begin{array}{ccc} \mathrm{colim}_{\Gamma}\, \mathbf{1} & \xrightarrow{\mathrm{id}} & \mathrm{colim}_{\Gamma}(\mathbf{1}) \\ \downarrow & \ulcorner & \downarrow \\ \mathbf{1} & \longrightarrow & \mathrm{colim}^{\mathbf{1}}_{\Gamma}(\mathbf{1}) \end{array}$$

Consequence: The type $\mathrm{colim}^{\mathbf{1}}_{\Gamma}(F)$ is also $(\mathcal{L}, \mathcal{R})$-connected.

Explicit colimit construction for higher groups

$(n\text{-connected}, n\text{-truncated})$ OFS $\Rightarrow$ Categories of *higher groups*

$$\mathcal{U}^{*}_{\geq k, \leq n+k} \quad := \quad \begin{array}{c} (k-1)\text{-}\textbf{\textit{connected}}, \\ (n+k)\text{-}\textit{truncated} \\ \textit{pointed types} \end{array}$$

inherit colimits from $\mathcal{U}^{*}$.

Pushout of coproducts

By the 3×3 *lemma*, transform our pushout construction $\mathcal{P}_F$ to a **new construction**:

$$\begin{array}{ccc}
\left(\bigvee_{i,j,g} \mathrm{pr}_1(F_i)\right) \vee \left(\bigvee_{i,j,g} \mathrm{pr}_1(F_i)\right) & \longrightarrow & \bigvee_i \mathrm{pr}_1(F_i) \\
\downarrow & & \downarrow \\
\bigvee_{i,j,g} \mathrm{pr}_1(F_i) & \longrightarrow & \mathrm{colim}_\Gamma^A(F)
\end{array}$$

Assume: A is $(\mathcal{L}, \mathcal{R})$-connected.

Note: Pushouts and coproducts preserve $(\mathcal{L}, \mathcal{R})$-connectedness.

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\downarrow & & \downarrow \\
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\end{array}$$

Assume: A is $(\mathcal{L}, \mathcal{R})$-connected.

Note: Pushouts and coproducts preserve $(\mathcal{L}, \mathcal{R})$-connectedness.

Consequence of new construction and note:

Full subcategory of $A/\mathcal{U}$ on $(\mathcal{L}, \mathcal{R})$-connected types has colimits.

Lemma

Let $G : \mathcal{U}^*_{\geq k, \leq n+k}$. The coslice $G/\mathcal{U}^*_{\geq k, \leq n+k}$ is a reflective subcategory (in a coherent sense) of $\mathrm{pr}_1(G)/\mathcal{U}_{\geq k, \leq n+k}$.

We just saw how to build colimits in $\mathrm{pr}_1(G)/\mathcal{U}_{\geq k, \leq n+k}$**.**

Example

The categories of *higher pointed abelian groups*

$$K(\mathbb{Z}, n)/\mathcal{U}^*_{\geq m, \leq n+m} \qquad (K(\mathbb{Z}, n) := \text{Eilenberg-MacLane space})$$

with $n, m > 0$ and $m < n$ inherit colimits from

$$\mathrm{pr}_1(K(\mathbb{Z}, n))/\mathcal{U}_{\geq m, \leq n+m}$$

Conclusion

Takeaway: A useful construction of colimits in $A/\mathcal{U}$

- **Technical report:**
  `https://doi.org/10.48550/arXiv.2411.15103`
- **Agda code:**
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Thanks!

References

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