On Left Adjoints Preserving Colimits in Homotopy Type Theory

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Outline

1. See whether left adjoints preserve colimits in **wild categories**.

   Wild category theory is a natural framework for synthetic homotopy theory.

2. Identify the bicategorical condition for the "standard" classical proof to work in wild category theory.

3. Apply this condition to a couple of important wild left adjoints: the **suspension** endofunctor and all **modalities**.

The standard proof

Consider a diagram $F : \mathcal{J} \to \mathcal{C}$ with a colimit $T := \mathrm{colim}_{\mathcal{J}}(F)$.

For every adjunction $L \dashv R$, we have a chain of (coherent) isos:

$$\begin{aligned} & \hom_{\mathcal{D}}(L(T), Y) \\ \cong\ & \hom_{\mathcal{C}}(T, R(Y)) \\ \cong\ & \lim_i(\hom_{\mathcal{C}}(F_i, R(Y))) \\ \cong\ & \lim_i(\hom_{\mathcal{D}}(L(F_i), Y)) \end{aligned}$$

This is *almost* the universal property of the colimit under the induced diagram $L(F)$.

Must check that this composite iso is the **canonical function**: the one appending a map $L(T) \to Y$ to the evident cocone on $L(T)$.

This requirement holds, for example, when the adjunction is formulated in

- 1-category theory
- bicategory theory
- $(\infty, 1)$-category theory

So, the standard proof that left adjoints preserve colimits (*LAPC*) works in these settings.

We want a way of proving *LAPC* in synthetic homotopy theory.

The standard proof seems to be the best candidate.[1]

Problem: The requirement may fail to hold for *wild adjunctions*.

This work addresses this problem.

[1] Section V.5 of *CWM* presents another well-known classical proof of *LAPC*, tailored to the unit-counit definition of adjunction, but it is even less amenable to wild categories than the standard proof.

Homotopy type theory (HoTT)

HoTT extends Martin-Löf type theory with

- the univalence axiom (which we don't really use here)
- higher inductive types (which we use for the applications)

The point of HoTT:

- Types are interpreted as *homotopy types*: topological spaces up to continuous deformation.
- HoTT has an interpretation in every $(\infty, 1)$-topos (Shulman).

Wild categories

Synthetic homotopy theory relies on categorical properties of types.

A universe $\mathcal{U}$ of types forms a **wild category**:

- has the same data as a 1-category
- except with untruncated hom-types

  Hom-types may no longer be sets (i.e., discrete spaces).

(It actually forms an $(\infty, 1)$-category, but this notion may not be definable in HoTT.)

Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be wild functors.

A **wild adjunction** $L \dashv R$ consists of

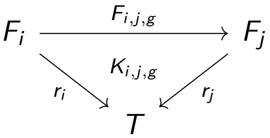
- a family of type equivalences

$$\alpha \ : \ \prod_{X:\mathrm{Ob}(\mathcal{D})} \prod_{A:\mathrm{Ob}(\mathcal{C})} \mathrm{hom}_{\mathcal{D}}(LA, X) \xrightarrow{\simeq} \mathrm{hom}_{\mathcal{C}}(A, RX)$$

- proofs V_c and V_d that α is natural in X and A, respectively.

Let $\mathcal{C}$ be a wild category, Γ be a directed graph, and F be a $\mathcal{C}$-valued diagram over Γ.[2]

Consider a cocone $\mathcal{K}$ under F with tip T:



where g is an edge in Γ from i to j and $K_{i,j,g}$ is a path witnessing that this triangle commutes.

Suppose that $\mathcal{K}$ is **colimiting**: for all $Y : \mathrm{Ob}(\mathcal{C})$, the following function is an equivalence:

$$- \circ \mathcal{K} \ : \ \hom_{\mathcal{C}}(T, Y) \to \underbrace{\lim_i(\hom_{\mathcal{C}}(F_i, Y))}_{\substack{\textit{type of cocones} \\ \textit{under } F \textit{ with tip } Y}}$$

[2]We only look at graph-indexed diagrams to avoid an infinite tower of coherences that may be undefinable in HoTT.

Replaying the standard proof

Consider a wild adjunction $L \dashv R$. We again have a chain of equivalences:

$$\begin{aligned} & \hom_{\mathcal{D}}(L(T), Y) \\ \simeq\ & \hom_{\mathcal{C}}(T, R(Y)) \\ \simeq\ & \lim_i(\hom_{\mathcal{C}}(F_i, R(Y))) \\ \simeq\ & \lim_i(\hom_{\mathcal{D}}(L(F_i), Y)) \end{aligned}$$

Problem: This composite need not be post-composition.

- They are equal (up to path) on the legs of a cocone.
- They may not agree on the triangle paths.

(We will see a counterexample.)

A sufficient condition

Our definition of *adjunction* is fine for 1-categories but not coherent enough for wild categories.

In particular, it says nothing about the interaction between

- the naturality square V_d of the adjunction
- the composition law $L_\circ$ of L, which shows up in the definition of $L(\mathcal{K})$.

We need a condition about this interaction to make the composite equivalence equal post-composition.

We say that L is 2-**coherent** if the following hexagon commutes for all suitable morphisms h_1, h_2, and h_3:

$$
\begin{array}{ccc}
(\alpha(h_1) \circ h_2) \circ h_3 & \overset{\mathsf{assoc}(\alpha(h_1), h_2, h_3)}{=\!=\!=\!=\!=\!=} & \alpha(h_1) \circ (h_2 \circ h_3) \\
\mathsf{ap}_{-\circ h_3}(V_d(h_2, h_1)) \Big\| & & \Big\| V_d(h_2 \circ h_3, h_1) \\
\alpha(h_1 \circ L(h_2)) \circ h_3 & & \alpha(h_1 \circ L(h_2 \circ h_3)) \\
V_d(h_3, h_1 \circ L(h_2)) \Big\| & & \Big\| \mathsf{ap}_{\alpha}(\mathsf{ap}_{h_1 \circ -}(L_{\circ}(h_2, h_3))) \\
\alpha((h_1 \circ L(h_2)) \circ L(h_3)) & \underset{\mathsf{ap}_{\alpha}(\mathsf{assoc}(h_1, L(h_2), L(h_3)))}{=\!=\!=\!=\!=\!=} & \alpha(h_1 \circ (L(h_2) \circ L(h_3)))
\end{array}
$$

In terms of a classical biadjunction, this hexagon is exactly the condition that the pseudotransformation

$$(\alpha, V_d) \ : \ \mathrm{hom}_{\mathcal{D}}(L(-), Y) \Rightarrow \mathrm{hom}_{\mathcal{C}}(-, R(Y))$$

respects composition of 1-cells.

Theorem

If L is 2-coherent, then $L(\mathcal{K})$ is colimiting in $\mathcal{D}$.

Counterexample

Define the wild category $\mathcal{E}$ by $\mathsf{Ob}(\mathcal{E}) :\equiv \mathbf{1}$ and $\mathsf{hom}_{\mathcal{E}}(*, *) :\equiv S^1$ with composition $\bullet$ coming from path composition on a loop space.

For each $\ell : \mathsf{hom}_{\mathcal{E}}(*, *)$, we have a nontrivial loop L_ℓ at ℓ.

Let the wild functor $\Lambda : \mathcal{E} \to \mathcal{E}$ be the identity on objects and morphisms, but let $\Lambda_\circ(\ell_1, \ell_2) :\equiv \mathsf{L}_{\ell_1 \bullet \ell_2}$.

Then Λ is not 2-coherent with respect to the evident wild adjunction $\Lambda \dashv \Lambda$: If $h_1 \equiv \mathsf{id}_*$, then we'd have to show that $\Lambda_\circ(h_2, h_3)$ is trivial, which is false.

Suspension is 2-coherent

Goal: Show that $\Sigma : \mathcal{U}^* \to \mathcal{U}^*$ is a 2-coherent left adjoint to Ω.

(The HIT $\Sigma(X)$ is the homotopy pushout of $\mathbf{1} \leftarrow X \to \mathbf{1}$.)

In this case, the *structure identity principle* turns 2-coherence into a (pointed) homotopy between pointed homotopies (*homotopy* :≡ pontwise path):

Definition

Let f_1 and f_2 be pointed maps and let $(H_1, \kappa_1), (H_2, \kappa_2) : f_1 \sim_* f_2$.

A *homotopy between* (H_1, κ_1) *and* (H_2, κ_2) consists of

- a homotopy $\tau : H_1 \sim H_2$
- a path $D_\tau : \kappa_1 =_\tau \kappa_2$ over τ.

In proving Σ is 2-coherent,

- τ: messy but manageable
- D_τ: infeasible.[3]

But the 2-coherence hexagon is about maps landing in a loop space, which is **homogeneous**.

A pointed type (X, x) is *homogeneous* if we have a pointed equivalence $(X, x) \simeq_* (X, y)$ for every $y : X$.

Lemma (yet another Cavallo's trick)

Let $f_1, f_2 : X_1 \to_* X_2$ with X_2 homogeneous. Let $(H_1, \kappa_1), (H_2, \kappa_2) : f_1 \sim_* f_2$. If $H_1 \sim H_2$, then (H_1, κ_1) and (H_2, κ_2) are homotopic.

Payoff for Σ: We ignore D_τ and are done!

[3] at least in traditional HoTT (the system we adopt)

Modalities are 2-coherent

A **modality** is a function $\bigcirc : \mathcal{U} \to \mathcal{U}$ along with maps $\eta_X : X \to \bigcirc X$ that forms a special kind of reflective subuniverse of $\mathcal{U}$ (hence a wild left adjoint to the forgetful functor).

Prime example: the n-truncation $\|-\|_n$

For every type A, we have an induced wild functor $\bigcirc^A : A/\mathcal{U} \to (A/\mathcal{U})_\bigcirc$ into the full subcategory of the coslice on *modal types* (types for which η is an equivalence).

Not hard to show that $\bigcirc^A$ is a 2-coherent left adjoint to the forgetful functor.

Corollary

The wild category $(A/\mathcal{U})_\bigcirc$ has all graph-indexed colimits.

Open questions

- A counterexample to *LAPC* for wild categories (not just to the standard proof)?

- A trick for showing that the smash product $X \wedge - : \mathcal{U}^* \to \mathcal{U}^*$ is 2-coherent for every $X : \mathcal{U}^*$?

- Do all reflective subuniverses of $\mathcal{U}$ satisfy 2-coherence?

  (I don't see how to generalize the proof for modalities.)

Conclusion

Takeaway: Wild left adjoints preserve colimits under a reasonable condition, which Σ and modalities satisfy.

Agda code: `https://github.com/PHart3/colimits-agda`

Thank you!