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Abstract

This is a brief introduction to elementary toposes. These play a central role in categorical semantics of dependent type theory (along with other areas of categorical logic). We assume knowledge of basic category theory, and our main source for this material is the nLab.

Let $\mathscr{C}$ be a category with finite limits. For any object $A \in \operatorname{ob}\mathscr{C}$, a *power object* of A is an object $\mathcal{P}(A)$ of $\mathscr{C}$ together with a monomorphism $\in_A \rightarrowtail A \times \mathcal{P}(A)$ such that for every monomorphism $f : C \to A \times D$ in $\mathscr{C}$, there is a unique pullback square of the form

$$\begin{array}{ccc} C & \longrightarrow & \in_A \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow \\ A \times D & \xrightarrow[\mathrm{id}_A \times \chi_f]{} & A \times \mathcal{P}(A) \end{array} .$$

We call χ_f the *classifying map* of f. If $A = 1$, then a power object of A is called a *subobject classifier*.

A category $\mathscr{E}$ is an *elementary topos* if it

- has finite limits,
- is cartesian closed, and
- has a subobject classifier $\texttt{true} : 1 \to \Omega$.

In this case, any global element $1 \to \Omega$ is called a *truth value*.

Let $\mathscr{C}$ be a category with finite limits. The mapping $X \in \operatorname{ob}\mathscr{C} \mapsto \texttt{Sub}(X)$, the subobject poset of X, induces a functor $\mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ sending a map $A \xrightarrow{f} B$ in $\mathscr{C}$ to the function $\texttt{Sub}(B) \xrightarrow{f^*(-)} \texttt{Sub}(A)$ of sets. The functor $\texttt{Sub}(-)$ is represented by Γ if and only if Γ is a subobject classifier of $\mathscr{C}$. By uniqueness of representing objects, it follows that a subobject classifier is unique up to isomorphism.

Theorem 1 (Fundamental theorem of topos theory). *If $\mathscr{E}$ is a topos and $X \in \operatorname{ob}\mathscr{E}$, then the overcategory $\mathscr{E}/X$ is a topos.*

Corollary 2. *Every topos is locally cartesian closed.*

Proposition 3. *A category $\mathscr{C}$ with finite limits is a topos if and only if every object of $\mathscr{C}$ has a power object.*

In particular, for any topos $\mathscr{E}$ and $A \in \operatorname{ob}\mathscr{E}$, the exponential object Ω^A is a power object of A. In this case, the power object functor $\Omega^{(-)} : \mathscr{E}^{\mathrm{op}} \to \mathscr{E}$ sends a map $X \xrightarrow{f} Y$ in $\mathscr{E}$ to the transpose of the composite

$$\Omega^Y \times X \xrightarrow{\mathrm{id}_{\Omega^B} \times f} \Omega^Y \times Y \xrightarrow{\mathrm{ev}_{Y,\Omega}} \Omega$$

under the adjunction $- \times X \vdash -^X$. We have a chain of natural isomorphisms

$$\mathscr{E}(X, \Omega^Y) \cong \mathscr{E}(X \times Y, \Omega) \cong \mathscr{E}(Y \times X, \Omega) \cong \mathscr{E}(Y, \Omega^X) \cong \mathscr{E}^{\mathrm{op}}(\Omega^X, Y),$$

which gives us an adjunction $\left(\Omega^{(-)}\right)^{\mathrm{op}} \vdash \Omega^{(-)}$. By an argument due to Paré, this adjunction is *monadic* in the sense that $\Omega^{(-)}$ reflects isomorphisms and preserves reflexive coequalizers, which implies that $\Omega^{(-)}$ creates limits. Since $\mathscr{E}$ has finite limits as a topos, it follows that $\mathscr{E}^{\mathrm{op}}$ has finite limits, i.e., $\mathscr{E}$ has finite *colimits*. In particular, $\mathscr{E}$ has an initial object 0.

Lemma 4. *The initial object of $\mathscr{E}$ is strict.*

Proof. Let $X \in \mathrm{ob}\,\mathscr{E}$ and $f : X \to 0$. We must show that f is an isomorphism. Notice that the map $0 \xrightarrow{\mathrm{id}_0} 0$ is both initial and terminal in the overcategory $\mathscr{E}/0$. The pullback functor $f^* : \mathscr{E}/0 \to \mathscr{E}/X$ has a left adjoint and thus preserves limits. Therefore, $g \coloneqq f^*(\mathrm{id}_0)$ is terminal in $\mathscr{E}/X$. We thus have an isomorphism $h : g \xrightarrow{\cong} \mathrm{id}_X$. Moreover, f^* has a right adjoint by Corollary 2 and thus preserves colimits. Hence g is also initial in $\mathscr{E}/X$. This means that $\mathrm{dom}(g) = 0$, and $f \circ g = \mathrm{id}_0$. The map h gives us an isomorphism $h : 0 \xrightarrow{\cong} X$ in $\mathscr{E}$ such that $g = \mathrm{id}_A \circ h$. This implies that $f = h^{-1}$, so that f is an isomorphism. □

Corollary 5. *For any $X \in \mathrm{ob}\,\mathscr{E}$, the unique map $0 \to X$ is monic.*

Notably, the classifying map of $0 \to 1$ is called *false*.

Proposition 6. *Suppose that $\Omega \xrightarrow{f} \Omega$ is monic. Then $f \circ f = \mathrm{id}_\Omega$ (so that f is an automorphism).*

Example 7.

1. The category **Set** is a *Boolean* topos, i.e., $\Omega \cong 1 \coprod 1$.
2. For any small category $\mathscr{C}$, the presheaf category $\widehat{\mathscr{C}} \coloneqq [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is a topos where the functor Ω sends $U \in \mathrm{ob}\,\mathscr{C}$ to the set $\mathtt{sieves}(U)$ of *sieves on* U, i.e., sets σ of morphisms over U such that for any morphisms $f : X \to Y$ and $g : Y \to U$ in $\mathscr{C}$,

   $$Y \xrightarrow{g} U \ \in \ \sigma \quad \Longrightarrow \quad X \xrightarrow{f} Y \xrightarrow{g} U \ \in \ \sigma.$$

   The action of Ω on morphisms in $\mathscr{C}$ is defined by

   $$V \xrightarrow{h} U \quad \mapsto \quad \sigma \mapsto \{f : X \to V \mid h \circ f \in \sigma,\ X \in \mathrm{ob}\,\mathscr{C}\}\,.$$

   The sieve on U generated by id_U is the top element $\mathtt{sieve}_{\mathtt{top}}(U)$ of $\mathtt{sieves}(U)$. We define $\mathtt{true} : 1 \to \Omega$ as the natural transformation with components

   $$\begin{aligned} \mathtt{true}(U) : \{*\} &\to \mathtt{sieves}(U) \\ * &\mapsto \mathtt{sieve}_{\mathtt{top}}(U). \end{aligned}$$

   For any monomorphism $\varphi : F \hookrightarrow G$ in $\widehat{\mathscr{C}}$, the classifying map of φ has components

   $$\begin{aligned} &\chi_\varphi(U) : G(U) \to \Omega(U) \\ &x \mapsto \{f : X \to U \mid G(f)(x) \in F(X),\ X \in \mathrm{ob}\,\mathscr{C}\}\,. \end{aligned}$$

   The subobject $\Omega_{\mathsf{dec}} \hookrightarrow \Omega$ consisting of decidable sieves classifies all monomorphisms $F \xrightarrow{\psi} G$ in $\widehat{\mathscr{C}}$ such that $\psi_A : F(A) \to G(A)$ has decidable image for every $A \in \mathrm{ob}\,\mathscr{C}$. Here, for any set T, a subset $S \subset T$ is decidable if and only if for any $x \in T$, the disjunction $x \in S \vee x \notin S$ is provable. If our metatheory includes LEM, then $\Omega_{\mathsf{dec}} = \Omega$.

Definition 8 (Heyting algebra). Let L be a bounded lattice. We say that L is a *Heyting algebra* if it has a binary operation $\Rightarrow : L \times L \to L$, called *implication*, such that

$$
\begin{aligned}
p \Rightarrow p &= 1 \\
p \wedge (p \Rightarrow q) &= p \wedge q \\
q \wedge (p \Rightarrow q) &= q \\
p \Rightarrow (q \wedge r) &= (p \Rightarrow q) \wedge (p \Rightarrow r)\,.
\end{aligned}
$$

For any topos $\mathscr{E}$ and $A \in \mathrm{ob}\,\mathscr{E}$, the poset $\mathtt{Sub}(A)$ is a Heyting algebra. As a result, $\mathtt{Sub}(A)$ is a model of intuitionistic propositional calculus. For example, the meet $\cap$ and join $\cup$ operation for $\mathtt{Sub}(A)$ are precisely the binary product and binary coproduct in $\mathtt{Sub}(A)$, respectively.

Proposition 9. *Let U_1 and U_2 be subobjects of A.*

1. *We have a pullback square*

$$
\begin{array}{ccc}
U_1 \cap U_2 & \longrightarrow & U_2 \\
\downarrow & \lrcorner & \downarrow \\
U_1 & \longrightarrow & A
\end{array}
$$

   in $\mathscr{E}$ consisting of monomorphisms.

2. *We have a pushout square*

$$
\begin{array}{ccccc}
U_1 \cap U_2 & \longrightarrow & U_2 & & \\
\downarrow & & \downarrow & \searrow & \\
U_1 & \longrightarrow & U_1 \cup U_2 & & \\
 & \searrow & & \overset{\alpha}{\dashrightarrow} & A
\end{array}
$$

   in $\mathscr{E}$ where α is a monomorphism.

Moreover, implication $\mathtt{Sub}(A) \times \mathtt{Sub}(A) \xrightarrow{\Rightarrow} \mathtt{Sub}(A)$ is defined by

$$U_1 \Rightarrow U_2 \ = \ \Pi_{U_1}(U_1 \cap U_2),$$

where Π denotes the dependent product.

Remark 10. A *Boolean algebra* is a Heyting algebra L where every $x \in L$ has a complement, i.e., an element $c_x \in L$ such that $x \vee c_x = 1$ and $x \wedge c_x = 0$. A topos $\mathscr{E}$ is Boolean if and only if $\mathtt{Sub}(A)$ is a Boolean algebra for all $A \in \mathrm{ob}\,\mathscr{E}$. In this case, $\mathtt{Sub}(A)$ satisfies LEM.

Let $\mathscr{E}$ be a topos and consider a map $\mathtt{El} : \widehat{U} \to U$ in $\mathscr{E}$. We say that a map $f : X \to Y$ in $\mathscr{E}$ is *U-small* if there exists a pullback square of the form

$$
\begin{array}{ccc}
X & \longrightarrow & \widehat{U} \\
{\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathtt{El}} \\
Y & \longrightarrow & U
\end{array}
. \qquad (*)
$$

Note that the class of U-small maps is closed under pullbacks. We say that $\mathtt{El}$ is a *universe in $\mathscr{E}$* if the class of U-small maps

(a) is closed under

- products,
- dependent sums,
- dependent products, and
- pullbacks of $1 \xrightarrow{\texttt{true}} \Omega$ and

(b) contains the unique map $\Omega \to 1$.

Condition (b) expresses that U is *impredicative.*

Although the square $(*)$ need not be unique, it will be when U has the structure of a univalent type-theoretic universe.

Example 11. The subobject classifier is a *predicative* universe as long as $\Omega \neq 1$, and the Ω-small maps are precisely the monomorphisms.

Remark 12. Closure under dependent sums is sometimes used as an alternative definition of *impredicative*, in which case Ω is impredicative. Unfortunately, both definitions appear in the type theory literature.