Voevodsky's Simplicial Model of Univalent Type Theory

Perry Hart

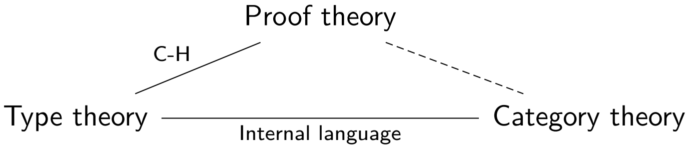
UPenn

April 24, 2020

Table of Contents

Table of Contents

Motivation



- Formalizing abstract homotopy theory
- Certified proofs of homotopical results
- Convenient notation for higher topos theory
- Expressive programming languages

Historical Development

- Awodey and Warren
- Voevodsky
- Harper

Table of Contents

Martin-Löf Type Theory

System of natural deduction consisting of *judgments*.

- **Context** "$x_1 : A_1, x_2 : A_2, \dots, x_n : A_n$ is a context,"

$$\mathtt{ctx}(x_1 : A_1, x_2 : A_2, \dots, x_n : A_n).$$

- **Typehood** "A is a type in context Γ,"

$$\Gamma \vdash A\,\mathtt{type}\,.$$

- **Typing declaration** "a is a term of type A in context Γ,"

$$\Gamma \vdash a : A.$$

- **Equality of terms** "a and b are definitionally equal terms of type A in context Γ,"

$$\Gamma \vdash a \equiv b : A.$$

Type Constructors

- Dependent product Π

$$\Pi\text{-FORM}\ \frac{\Gamma \vdash A\,\mathtt{type} \qquad \Gamma, x : A \vdash B(x)\,\mathtt{type}}{\Gamma \vdash \Pi_{x:A} B(x)\,\mathtt{type}}$$

$$\Pi\text{-INTRO}\ \frac{\Gamma, x : A \vdash B(x)\,\mathtt{type} \qquad \Gamma, x : A \vdash b(x) : B(x)}{\Gamma \vdash \lambda(x : A).b(x) : \Pi_{x:A} B(x)}$$

$$\Pi\text{-ELIM}\ \frac{\Gamma \vdash f : \Pi_{x:A} B(x) \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{app}(f, a) : B[a/x]}$$

- Dependent sum Σ
- Tarski universe U

$$\frac{}{\vdash \mathsf{U}\,\mathtt{type}} \qquad\qquad \frac{}{x : \mathsf{U} \vdash \mathsf{el}(x)\,\mathtt{type}}$$

Curry-Howard Isomorphism

Encoding of constructive logic.

FOL (with bounded quantifiers)	Set theory	Type theory ($\mathbb{T}$)
Proposition	Set	Well-formed type
Proof	Element	Inhabitant
$\neg A$	A^c	$A \to \mathbf{0}$
$A \wedge B$	$A \times B$	$A \times B$
$A \to B$	B^A	$A \to B$
$\forall_{x:A} B(x)$	$\prod_{x \in A} B(x)$	$\Pi_{x:A} B(x)$
$\exists_{x:A} B(x)$	$\coprod_{x \in A} B(x)$	$\Sigma_{x:A} B(x)$
$\top$	$\{0\}$	$\mathbf{1}$
$\bot$	$\emptyset$	$\mathbf{0}$

Table of Contents

Identity Types

- *Propositional* equality $\mathsf{Id}_A(a, b)$
- Terms as paths $a \rightsquigarrow b$ in A
- Constant path $\mathsf{refl}(A, a)$ at a
- Path induction

$$\frac{\begin{array}{c}\Gamma, x : A, y : A, p : \mathsf{Id}_A(x, y) \vdash C(x, y, p)\ \mathtt{type} \\ \Gamma, z : A \vdash c(z) : C[z, z, \mathsf{refl}(A, z)/x, y, p]\end{array}}{\Gamma, x : A, y : A, p : \mathsf{Id}_A(x, y) \vdash \mathsf{J}(x.y.p.C, z.c, x, y, p) : C(x, y, p)}$$

Univalence Axiom

- A function $A \to B$ with both a section and a retraction is an *equivalence* $\xrightarrow{\simeq}$.

Univalence Axiom

- A function $A \to B$ with both a section and a retraction is an *equivalence* $\xrightarrow{\simeq}$.
- Construct a function

$$\mathtt{idtoequiv}_{x:A;B(x)}(x, y) : (x \rightsquigarrow_A y) \to (B(x) \simeq B(y))$$

using path induction.

Univalence Axiom

- A function $A \to B$ with both a section and a retraction is an *equivalence* $\xrightarrow{\simeq}$.
- Construct a function

$$\mathtt{idtoequiv}_{x:A;B(x)}(x,y) : (x \rightsquigarrow_A y) \to (B(x) \simeq B(y))$$

  using path induction.
- For any two types A and B, *declare* the function

$$\mathtt{idtoequiv}_{x:\mathsf{U};\mathsf{el}(x)}(A,B) : (A \rightsquigarrow_{\mathsf{U}} B) \to (\mathsf{el}(A) \simeq \mathsf{el}(B))$$

  to be an equivalence.

Univalence Axiom

- A function $A \to B$ with both a section and a retraction is an *equivalence* $\xrightarrow{\simeq}$.
- Construct a function

$$\mathtt{idtoequiv}_{x:A;B(x)}(x,y) : (x \rightsquigarrow_A y) \to (B(x) \simeq B(y))$$

  using path induction.
- For any two types A and B, *declare* the function

$$\mathtt{idtoequiv}_{x:\mathsf{U};\mathsf{el}(x)}(A,B) : (A \rightsquigarrow_{\mathsf{U}} B) \to (\mathsf{el}(A) \simeq \mathsf{el}(B))$$

  to be an equivalence.
- Obtain an inverse function of the form

$$(\mathsf{el}(A) \simeq \mathsf{el}(B)) \to (A \rightsquigarrow_{\mathsf{U}} B)\,.$$

Intuition

- Homotopy equivalent spaces make up first and last frame of a movie.
- Isomorphic objects are genuinely equal.

Table of Contents

Foundations

- Consistency of $\mathrm{MLDTT} + \mathrm{Univ}$
- Categorical semantics
- Soundness and completeness
- Non-constructive metatheory

Interpretation of syntax

Challenge

Construct an interpretation of syntax that commutes *strictly* with substitution and all type constructors.

- Contexts of length n as objects of degree n
- Dependent types $x : A \vdash B(x)$ as fibrations over A
- Terms of type B as sections of such fibrations
- Substitution as pullback

Contextual Categories

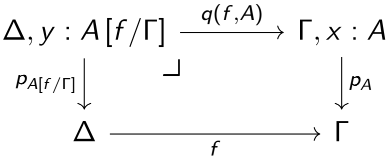
1. A terminal object 1
2. An $\mathbb{N}$-grading $\coprod_{n\in\mathbb{N}} \mathrm{Ob}_n \mathscr{C}$ of $\mathrm{Ob}\,\mathscr{C}$, $\mathrm{Ob}_0(\mathscr{C}) = \{1\}$
3. A function $\mathrm{ft}_n : \mathrm{Ob}_{n+1}\mathscr{C} \to \mathrm{Ob}_n \mathscr{C}$ for each n
4. Canonical projections $p_A : X \to \mathrm{ft}_n(X)$ + pullbacks of p_A

$$(fg)^* X = g^* (f^* X)$$

Example

The syntactic category $\mathscr{C}(\mathbb{T})$ encodes the syntax of $\mathbb{T}$.

- Contexts as objects
- Certain sequences of derivable typing declarations as morphisms



Universe Categories

A *universe in* $(\mathscr{C}, 1)$ is an object U equipped with a morphism $p : \widehat{U} \to U$ and, for each map $f : X \to U$, a distinguished pullback square

$$\begin{array}{ccc} (X; f) & \xrightarrow{Q(f)} & \widehat{U} \\ {\scriptstyle P_{X,f}}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & U \end{array} .$$

Intuitively, p corresponds to the dependent type $x : \mathrm{U} \vdash \mathrm{el}(x)\,\mathtt{type}$.

Proposition

Every universe category $(\mathscr{C}, U, 1)$ yields a contextual category $\mathscr{C}_U$.

Logical Structure on $\mathscr{C}_U$

MLDTT	LCCC
Dependent product	Dependent product
Dependent sum	Dependent sum
Empty type	Initial object
Unit type	Terminal object
Boolean type	$1 \coprod 1$
Identity type	Path space object
Universe type	Internal universe

A $\mathbb{T}$-*structure* is a contextual category with all of these structures.

A *logical contextual functor* is a functor of $\mathbb{T}$-structures preserving all properties on the nose.

Initiality Conjecture

For any $\mathbb{T}$-structure $\mathscr{C}$, there exists a unique logical contextual functor

$$\mathscr{C}(\mathbb{T}) \to \mathscr{C}.$$

In other words, any $\mathbb{T}$-structure correctly interprets the syntax of $\mathbb{T}$.

Corollary

- Our type theory is sound.
- Any theorem of our type theory holds in any categorical model of $\mathbb{T}$.

Blueprint

1. Find a Kan complex (closed type) U in **sSet** endowed with suitable logical structure.
2. Find a univalent Kan fibration in the induced contextual category $\mathbf{sSet}_U$.

Table of Contents

Fibrant Universes

Let κ be a (strongly) inaccessible cardinal.

Let $f : X \to Y$ be a map of simplicial sets.

1. We say that f is *well-ordered* if it is equipped with a well-ordering of $Y_x := f_n^{-1}(x)$ for each simplex $x \in X_n$.
2. We say that f is *κ-small* if $|Y_x| < \kappa$ for every simplex x.

Fibrant Universes

Let κ be a (strongly) inaccessible cardinal.

Let $f : X \to Y$ be a map of simplicial sets.

1. We say that f is *well-ordered* if it is equipped with a well-ordering of $Y_x := f_n^{-1}(x)$ for each simplex $x \in X_n$.
2. We say that f is κ*-small* if $|Y_x| < \kappa$ for every simplex x.

We have a presheaf

$$\mathcal{U}_\kappa(-) : \mathbf{sSet}^{\mathrm{op}} \to \mathbf{Set}$$

with $\mathcal{U}_\kappa(X)$ consisting of all iso. classes of κ-small well-ordered Kan fibrations over X.

This acts on morphisms by pullback.

Consider the simplicial set

$$\mathrm{U}_\kappa \coloneqq \mathcal{U}_\kappa \circ \mathcal{Y}^{\mathsf{op}} : \Delta^{\mathsf{op}} \to \mathbf{Set}$$

where $\mathcal{Y} : \Delta \to \mathbf{sSet}$ denotes the Yoneda embedding.

The Yoneda lemma induces an isomorphism

$$\mathcal{U}_\kappa(-) \xrightarrow{\cong} \mathsf{Hom}_{\mathbf{sSet}}(-, \mathrm{U}_\kappa),$$

from which we recover a certain Kan fibration

$$p_\kappa : \widehat{\mathrm{U}}_\kappa \to \mathrm{U}_\kappa.$$

For any κ-small well-ordered Kan fibration $f : Y \to X$, there exists a unique pullback square of the form

$$\begin{array}{ccc} Y & \longrightarrow & \widehat{\mathrm{U}}_\kappa \\ {\scriptstyle f}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p_\kappa} \\ X & \underset{\ulcorner f \urcorner}{\longrightarrow} & \mathrm{U}_\kappa \end{array} .$$

- This exhibits U_κ as a universe in **sSet**.
- It also gives rise to our desired logical structure on U_κ.

For any κ-small well-ordered Kan fibration $f : Y \to X$, there exists a unique pullback square of the form

$$\begin{array}{ccc} Y & \longrightarrow & \widehat{\mathrm{U}}_\kappa \\ {\scriptstyle f}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p_\kappa} \\ X & \underset{\ulcorner f \urcorner}{\longrightarrow} & \mathrm{U}_\kappa \end{array} .$$

- This exhibits U_κ as a universe in **sSet**.
- It also gives rise to our desired logical structure on U_κ.

Theorem

1. U_κ is a Kan complex.
2. For any inaccessible cardinal $\lambda < \kappa$, U_λ is an internal universe in U_κ.

Theorem

The Kan fibration $p_\kappa : \widehat{\mathrm{U}}_\kappa \to \mathrm{U}_\kappa$ satisfies Univ.

Corollary

For any inaccessible cardinal $\lambda < \kappa$, the contextual category $\mathbf{sSet}_{\mathrm{U}_\lambda}$ models $\mathrm{MLDTT} + \mathrm{Univ}$.

Generalizations

- Constructive metatheory (Gambino and Henry 2019)
- Grothendieck ∞-toposes (Shulman 2019)
- Elementary ∞-toposes (in progress)