Perry Hart K-theory reading seminar UPenn October 3, 2018

Abstract

We explore universal properties in category theory, especially limits and colimits. The main sources for this talk are the following.

- nLab
- John Rognes's Lecture Notes on Algebraic K-Theory, Ch. 4
- Peter Johnstone's lecture notes for "Category Theory" (Mathematical Tripos Part III, Michaelmas 2015), Ch. 4
- Steve Awodey's Category Theory, Sect. 5.6

1 Universal arrows

Definition 1.1. Let \mathscr{C} be a category and $X \in \operatorname{ob} \mathscr{C}$.

- 1. We say that X is *initial* if for each $Y \in ob \mathcal{C}$, there is a unique morphism $f: X \to Y$.
- 2. We say that X is terminal if for each $Z \in ob \mathscr{C}$, there is a unique morphism $g: Z \to X$.

Either condition is called a *universal property* of X.

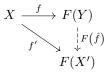
Any property P of \mathscr{C} (expressible in our the language of category theory) has a dual property P^{op} of \mathscr{C}^{op} obtained by interchanging the source and target of any arrow as well as the order of any composition in the sentence expressing P. Then P is true of \mathscr{C} iff P^{op} is true of \mathscr{C}^{op} .

Example 1.2. Being initial and being terminal are dual properties.

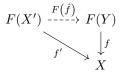
Lemma 1.3. Any initial object of \mathcal{C} is unique up to unique isomorphism. The same holds for any terminal object of \mathcal{C} .

Proof sketch. Let X and X' be two initial objects. The two unique morphisms $X \to X'$ and $X' \to X$ form an isomorphism between X and X', which must be unique by definition of an initial object. Apply duality to this argument to deduce that any terminal object is unique.

We can think of a universal property as follows. Let $F : \mathscr{D} \to \mathscr{C}$ be a functor and $X \in ob \mathscr{C}$. A universal arrow from X to F is an ordered pair (Y, f) with $Y \in ob \mathscr{D}$ and $f : X \to F(Y)$ a morphism in \mathscr{C} such that for any $X' \in ob \mathscr{D}$ and morphism $f' : X \to F(X')$ in \mathscr{C} , there exists a unique morphism $\hat{f} : Y \to X'$ of \mathscr{D} such that $F(\hat{f}) \circ f = f'$.



Dually, a universal arrow from F to X is an ordered pair (Y, f) with $Y \in ob \mathscr{D}$ and $f : F(Y) \to X$ of \mathscr{C} with the property that for any $X' \in ob \mathscr{D}$ and morphism $f' : F(X') \to X$, there exists a unique morphism $\hat{f} : X' \to Y$ such that $f' = f \circ F(\hat{f})$.



To see why this notion of universality agrees with our original one, we first generalize the notion of an arrow category.

Definition 1.4. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor and $Y \in \operatorname{ob} \mathscr{D}$.

- 1. The slice or left fiber category, denoted by (F/Y) or $(F \downarrow Y)$, has as objects pairs (X, f) where $X \in ob \mathscr{C}$ and $f \in \mathscr{D}(F(X), Y)$ and as morphisms from $f : F(X) \to Y$ to $f' : F(X') \to Y$ morphisms $g : X \to X'$ in \mathscr{C} such that $f = f' \circ F(g)$.
- 2. The coslice or right fiber category, denoted by (Y/F) or $(Y \downarrow F)$, has as objects pairs (X, f) where $X \in ob \mathscr{C}$ and $f \in \mathscr{D}(Y, F(X))$ and as morphisms from $f: Y \to F(X)$ to $f': Y \to F(X')$ morphisms $g: X \to X'$ in \mathscr{C} such that $f' = F(g) \circ f$.

Consider the opposite functor $F^{\text{op}} : \mathscr{C}^{\text{op}} \to \mathscr{D}^{\text{op}}$ of a functor $F : \mathscr{C} \to \mathscr{D}$. For any $Y \in \text{ob} \mathscr{D}$, we have that $(Y/F)^{\text{op}} = F^{\text{op}}/Y$. Thus, the left and right fiber categories are dual in the sense that P(Y, F) is true of any right fiber category Y/F iff $P^{\text{op}}(Y, F)$ is true of any left fiber category F/Y.

Proposition 1.5. Let $F : \mathcal{D} \to \mathcal{C}$ be a functor and $x \in ob \mathcal{C}$. Then $u : x \to Fr$ is a universal arrow from x to F iff it is an initial object of $(x \downarrow F)$. Dually, $u' : Fr' \to x$ is a universal arrow from F to x iff it is a terminal object of $(F \downarrow x)$.

Proof. Suppose that u is universal and $f: x \to Fy$ is another object of $(x \downarrow F)$. Then there exists a unique map $\hat{f}: r \to y$ such that $F(\hat{f}) \circ u = f$. Thus, $F(\hat{f})$ is a unique map in the coslice from u to f.

Conversely, suppose that u is initial. Then for any object $f: x \to Fy$ of $(x \downarrow F)$, there exists a unique arrow $Sg: Fr \to Fy$ such that $Sg \circ u = f$. Thus, taking $\hat{f} = g$ makes u a universal arrow.

Corollary 1.6. Any two universal arrows from x to F can be canonically identified by Lemma 1.3.

2 Colimits

Let \mathscr{C} be a category.

Definition 2.1. We say that the initial object 0 of \mathscr{C} is *strict* if every morphism of the form $X \to 0$ is an isomorphism.

Proposition 2.2. Suppose that 0 is strict. For any $Z \in ob \mathcal{C}$, the unique map $0 \to Z$ is monic.

Proof. Let $A \in ob \mathscr{C}$ and consider any two maps $k : A \to 0$ and $h : A \to 0$. These are isomorphisms, and both k^{-1} and h^{-1} equal the unique map $0 \to A$. This means that k = h. It follows immediately that $0 \to Z$ is monic.

Definition 2.3. A zero object of \mathscr{C} is an object that is both initial and terminal.

By Lemma 1.3, any zero object is unique up to unique isomorphism.

Example 2.4. The initial object of **Set** is \emptyset , and the terminal objects are precisely the singleton sets. Hence there is no zero object. Moreover, there is no initial or terminal object in iso(**Set**).

For any $X \in ob \mathscr{C}$, the *undercategory* X/\mathscr{C} has as objects morphisms in \mathscr{C} of the form $i: X \to Y$. Given $i: X \to Y$ and $i': X \to Y'$ in $ob X/\mathscr{C}$, define a morphism from i to i' as a morphism $f: Y \to Y'$ where



commutes. (We call *i* the *structure morphism*.) Both composition and the set of identity maps are inherited directly from \mathscr{C} .

Likewise, for any $x \in ob \mathcal{C}$, the *overcategory* \mathcal{C}/X has as objects morphisms in \mathcal{C} of the form $i: Y \to X$. Given $i: Y \to X$ and $i': Y' \to X$ in $ob \mathcal{C}/X$, define a morphism from i to i' as a morphism $f: Y \to Y'$ where



commutes. Again, both composition and the set of identity maps are inherited directly from \mathscr{C} .

Remark 2.5. If $X \in ob \mathscr{C}$, then $(X/\mathscr{C})^{op} = \mathscr{C}^{op}/X$. Thus, the under- and overcategory are dual in the sense that $P(X, \mathscr{C})$ is true of any undercategory X/\mathscr{C} iff $P^{op}(X, \mathscr{C})$ is true of any overcategory \mathscr{C}/X .

Lemma 2.6. For any $X \in \mathcal{C}$, the identity morphism on X is an initial object X/\mathcal{C} . Dually, it is a terminal object in \mathcal{C}/X .

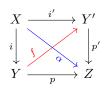
Proof. Any morphism $i: X \to Y$ is itself the unique morphism from Id_X to i.

Lemma 2.7. Let X be an initial object of \mathscr{C} . The identity morphism on X is a zero object \mathscr{C}/X . Dually, if $Y \in ob \mathscr{C}$ is terminal, then Id_Y is a zero object in Y/\mathscr{C} .

Proof. We already know that Id_X is terminal. If $p: Y \to X$ is an object in \mathscr{C}/X , then there is a unique morphism $f: X \to Y$. Then $f \circ p$ must equal Id_X .

Example 2.8. For any set x, consider the pointed set $X \coloneqq \{x\}$. Let \mathbf{Set}_* denote the category of pointed sets with basepoint-preserving functions. Since $\mathbf{Set}_* \cong X/\mathbf{Set}$, it follows that X is a zero object in \mathbf{Set}_* .

Given a morphism $\alpha : X \to Z$ in \mathscr{C} , define the *under-and-overcategory* $(X/\mathscr{C}/Z)_{\alpha}$ as having triples (Y, i, p) as objects where $i : X \to Y$ and $p : Y \to Z$ are morphisms in \mathscr{C} such that $p \circ i = \alpha$. Define the set of morphisms from (Y, i, p) to (Y', u', p') as the set of morphisms $f : Y \to Y'$ such that

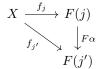


commutes. If $\alpha = \text{Id}_X$, then we call $(X/\mathscr{C}/X)_{\text{Id}_X}$ the category of *retractive* objects over X, with each triple (Y, i, p) being a retraction of Y onto X.

Example 2.9. If $F : \mathscr{C} \to \mathscr{C}$ is the identity functor, then the undercategory Y/\mathscr{C} equals the right fiber category Y/F, and the overcategory \mathscr{C}/Y equals the left fiber category F/Y.

Let \mathscr{J} and \mathscr{C} be categories. A diagram of shape \mathscr{J} in \mathscr{C} is a functor $F : \mathscr{J} \to \mathscr{C}$.

Definition 2.10. For any functor $F : \mathscr{J} \to \mathscr{C}$ and $X \in ob \mathscr{C}$, a *cone over* F consists of an *apex* $X \in ob \mathscr{C}$ and *legs* $f_j : X \to F(j)$ $(j \in ob \mathscr{J})$ such that for any morphism $\alpha : j \to j'$, the triangle



commutes.

A cone over F is just a natural transformation $\Delta_{\mathscr{J}}X \Rightarrow F$ where $\Delta_{\mathscr{J}}X$ denotes the constant functor on \mathscr{J} at X. (If \mathscr{J} is small, then $\Delta_{\mathscr{J}}$ is a functor from \mathscr{C} to $\operatorname{Fun}(\mathscr{J}, \mathscr{C})$.) The notion of *cocone under* Fis dual to that of cone.

Definition 2.11 (Colimit). Let \mathscr{C} and \mathscr{D} be categories and $g : Y \to Z$ be a morphism in \mathscr{D} . Let $\Delta_{\mathscr{C}}g : \Delta_{\mathscr{C}}Y \Rightarrow \Delta_{\mathscr{C}}Z$ be the natural transformation where every component is exactly g.

- 1. A colimit colim $_{\mathscr{C}}F$ of a functor $F: \mathscr{C} \to \mathscr{D}$ is an object Y of \mathscr{D} together with a natural transformation $i: F \Rightarrow \Delta_{\mathscr{C}}Y$ such that for any $Z \in \operatorname{ob} \mathscr{D}$ and any natural transformation $j: F \Rightarrow \Delta_{\mathscr{C}}Z$, there is a unique morphism $g: Y \to Z$ such that $j = \Delta_{\mathscr{C}}g \circ i$.
- 2. We say that \mathscr{D} admits/has \mathscr{C} -shaped colimits if each functor $G: \mathscr{C} \to \mathscr{D}$ has a colimit.
- 3. We say that \mathscr{D} is *cocomplete* if it admits \mathscr{C} -shaped colimits whenever \mathscr{C} is small.

If \mathscr{C} is small, then a colimit of $F : \mathscr{C} \to \mathscr{D}$ is just an initial object in the right fiber category $F/\Delta_{\mathscr{C}}$, which has as objects pairs $(Z, j : F \to \Delta_{\mathscr{C}} Z)$ and as morphisms from (Y, i) to (Z, j) morphisms $g : Y \to Z$ in \mathscr{D} such that $\Delta_{\mathscr{C}} g \circ i = j$.

Example 2.12. If \mathscr{C} is the empty category, then the functor $F : \mathscr{C} \to \mathscr{D}$ satisfies $F/\Delta_{\mathscr{C}} \cong \mathscr{D}$, so that the colimit of F is exactly the initial object of \mathscr{D} .

Proposition 2.13. There is a natural bijection $\mathscr{D}(Y,Z) \cong \operatorname{Fun}(\mathscr{C},\mathscr{D})(F,\Delta Z)$ if and only if $Y = \operatorname{colim}_{\mathscr{C}} F$.

Lemma 2.14. Any colimit of a functor is unique up to unique isomorphism.

Proof. When \mathscr{C} is small, this follows immediately from Lemma 1.3. Notice, however, that our proof of Lemma 1.3 does *not* require that \mathscr{C} be locally small (a property which Rognes stipulates of any category). \Box

Remark 2.15. Assume that \mathscr{D} has \mathscr{C} -shaped colimits and that \mathscr{C} is small. Then a (possibly global) choice function $\operatorname{colim}_{\mathscr{C}} : \operatorname{Fun}(\mathscr{C}, \mathscr{D}) \to \mathscr{D}$ given by choosing a colimit for each functor induces a functor that is left adjoint to the constant diagram functor $\Delta_{\mathscr{C}} : \mathscr{D} \to \operatorname{Fun}(\mathscr{C}, \mathscr{D})$. Indeed, for any functor $F : \mathscr{C} \to \mathscr{D}$, there is a bijection $\mathscr{D}(\operatorname{colim}_{\mathscr{C}} F, Z) \cong \operatorname{Fun}(\mathscr{C}, \mathscr{D})(F, \Delta_{\mathscr{C}} Z)$. **Definition 2.16 (Limit).** The *limit* of the functor $F : \mathscr{C} \to \mathscr{D}$ is the colimit of $F^{\text{op}} : \mathscr{C}^{\text{op}} \to \mathscr{D}^{\text{op}}$.

Explicitly, a limit for $F : \mathscr{C} \to \mathscr{D}$ is an object Z of \mathscr{D} along with a natural transformation $p : \Delta_{\mathscr{C}} Z \Rightarrow F$ such that for any $Y \in \operatorname{ob} \mathscr{D}$ and any natural transformation $q : \Delta_{\mathscr{C}} Y \Rightarrow F$, there is a unique morphism $g : Y \to Z$ such that $q = p \circ \Delta_{\mathscr{C}} g$.

Note that the colimit of a functor F is exactly the limit of F^{op} . Hence *limit* and *colimit* are dual properties, and all of our results for colimits can be dualized for limits.

Definition 2.17. Let $F: \mathscr{C} \to \mathscr{D}$ be a functor. We say that F creates limits if

- for any functor $G: \mathscr{J} \to \mathscr{C}, G$ has a limit when $F \circ G$ has a limit;
- F preserves all limits of G; and
- F reflects all limits of G.

Definition 2.18 (Product). Let \mathscr{J} be a discrete small category. Consider a diagram $\{A_i\}_{i \in ob} \mathscr{J}$ of shape \mathscr{J} .

- 1. The limit of $\{A_i\}_i$ is called the *product* $\prod_i A_i$, equipped with projections $\pi_i : \prod_i A_i \to A_i$ such that for every $f_i : U \to A_i$ there exists a unique map $f \coloneqq (f_i) : U \to \prod_i A_i$ satisfying $\pi_i \circ f = f_i$.
- 2. The colimit of $\{A_i\}_i$ is called the *coproduct* $\coprod_i A_i$, equipped with inclusions $u_i : A_i \to \coprod_i A_i$ such that for any $f_i : A_i \to Y$, there exists a unique map $f := (f_i) : \coprod_i A_i \to Y$ satisfying $f_i = f \circ u_i$.

Familiar examples of limits include cartesian products and direct products, whereas familiar examples of colimits include disjoint unions and free products.

Example 2.19.

(1) Consider any small diagram $F: \mathscr{J} \to \mathbf{Set}$. On the one hand,

$$\operatorname{colim}_{j} F_{j} \cong \left(\coprod_{j \in \operatorname{ob} \mathscr{J}} F_{j} \right) \middle/ \sim$$

where \sim is the smallest equivalence relation such that $F_j \ni f_j \sim f_{j'} \in F_{j'}$ whenever $F(\psi)(f_j) = f_{j'}$ for some $\psi : j \to j'$.

On the other hand,

$$\lim_{j} F_{j} \cong \left\{ (f_{j})_{j} \in \prod_{j \in ob \mathscr{J}} F_{j} \mid \forall \psi : j \to j' \text{ in } \mathscr{J}, \ F(\psi)(f_{j}) = f_{j'} \right\}.$$

This shows that **Set** is both complete and cocomplete.

(2) Let A be any set. Define the *cumulative hierarchy* $V_n(A)$ of rank $n < \omega$ over A along with a countable sequence

$$V_0 \xrightarrow{v_0} V_1 \xrightarrow{v_1} V_2 \longrightarrow \cdots \longrightarrow V_n \xrightarrow{v_n} V_{n+1} \longrightarrow \cdots$$

of maps recursively by

$$V_{0}(A) = A$$

$$V_{n+1}(A) = A \coprod \mathcal{P}(V_{n}(A))$$

$$v_{0} : A \hookrightarrow A \coprod \mathcal{P}(A), \qquad a \mapsto a$$

$$v_{n+1} : A \coprod \mathcal{P}(V_{n}(A)) \to A \coprod \mathcal{P}(V_{n+1}(A)), \quad (\mathrm{Id}_{A}, \mathcal{P}(V_{n}(A))).$$

Let $V_{\omega}(A) = \operatorname{colim}_{n < \omega} V_n(A)$, which exists by part (1). Then $V_{\omega}(\emptyset)$ is exactly the set of all hereditarily finite sets. To see that $V_{\omega}(-)$ is a functor **Set** \to **Set**, let $f : A \to B$ be a function. Then we can build a cocone

$$V_{0}(A) \xrightarrow{v_{0}} V_{1}(A) \longrightarrow \cdots \longrightarrow V_{n}(A) \xrightarrow{v_{n}} V_{n+1}(A)$$

$$f_{0} \equiv f \downarrow \qquad \qquad \downarrow (f, \mathcal{P}(f)) \qquad \qquad \downarrow (f, \mathcal{P}(f_{n-1})) \qquad \downarrow (f, \mathcal{P}(f_{n}))$$

$$V_{0}(B) \longrightarrow V_{1}(B) \longrightarrow \cdots \longrightarrow V_{n}(B) \longrightarrow V_{n+1}(B) \longrightarrow V_{\omega}(B)$$

under $\{V_n(A)\}_n$ recursively. By the universal property of colimits, there exists a unique map $V_{\omega}(A) \rightarrow V_{\omega}(B)$, so that $V_{\omega}(-)$ is functorial.

Let \mathscr{J} be a category of the form $\bullet \rightrightarrows \bullet$, known as a parallel pair. If the two maps have a common section, then we say that the pair is *reflexive*. A diagram D of shape \mathscr{J} looks like $A \stackrel{f}{\rightrightarrows} B$. A cone over D with apex C and legs $f_1: C \to A$ and $f_2: C \to B$ satisfies $f \circ f_1 = f_2 = g \circ f_1$.

Definition 2.20 (Equalizer).

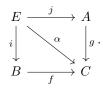
- 1. If such an object C together with f_1 is the limit of D, then we call it the *equalizer* of f and g.
- 2. The colimit of D is the *coequalizer* of f and g.

Example 2.21. The equalizer in **Set** of $f, g : X \to Y$ equals the subset $X' := \{x \in X : f(x) = g(x)\}$ together with the inclusion function $X' \hookrightarrow X$.

The coequalizer of (f, g) is precisely $Y \not\sim$ together with the quotient map on B where \sim is the smallest equivalence relation under which $f(x) \sim g(x)$ for every x.

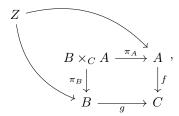
It is easy to check that any equalizer $f: C \to A$ is monic. Further, if f is split epic, i.e., has a section $g: A \to C$, then f is an isomorphism. For, in this case, $f \circ (g \circ f) = \operatorname{Id}_A \circ f = f \circ \operatorname{Id}_C$. As f is monic, we have that $g \circ f = \operatorname{Id}_C$, so that g is an inverse of f.

Next, let \mathscr{J} be a category of the form $\bullet \to \bullet \leftarrow \bullet$, known as a cospan. A diagram of this shape looks like $B \xrightarrow{f} C \xleftarrow{g} A$, and a cone over this diagram looks like



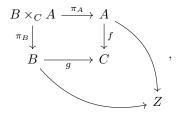
Definition 2.22 (Pullback). If such an object *E* together with *i* and *j* is the limit of this diagram, then we call it the *pullback* of *f* and *g*, denoted by $B \times_C A$.

The universal property of a pullback square states that for any commutative diagram of the form



there is a unique mediating map $Z \to B \times_C A$ fitting into it.

If we perform a dual construction for \mathscr{J}^{op} , then the colimit of the resulting diagram is called the *pushout*, denoted by $B \cup_C A$. The universal property of a pullback square states that for any commutative diagram of the form



there is a unique mediating map $B \cup_C A \to Z$ fitting into it.

Example 2.23.

- 1. The pullback in **Set** of $f: X \to Z$ and $g: Y \to Z$ is precisely $\{(x, y) \in X \times Y : f(x) = g(y)\}$, called the *fibered product* of X and Y over Z.
- 2. The pushout in **Set** of $f : Z \to X$ and $g : Z \to Y$ is precisely the quotient of $X \coprod Y$ by the equivalence relation \sim generated by the formula $(\forall z \in Z) (f(z) \sim g(z))$. We call $X \coprod Y / \sim$ the fibered sum of X and Y under Z.

Example 2.24. Let \mathbf{FinSet}_{mono} denote the category of all finite sets with injective functions as arrows. The category of *nominal sets* consists of all pullback-preserving functors $\mathbf{FinSet}_{mono} \rightarrow \mathbf{Set}$ with natural transformations as arrows. These correctly encode the syntax of functional programming languages modulo renaming of bound variables (which is necessary for implementing substitution).

Proposition 2.25. The pullback of a monomorphism in a category \mathscr{C} is also a monomorphism.

Proof. Consider any pullback square

$$\begin{array}{ccc} B \times_C A & \xrightarrow{\pi_2} & A \\ \pi_1 & & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

in \mathscr{C} where f is monic. We must show that π_1 is monic. Let $h_1, h_2: B' \to B \times_C A$ be morphisms in \mathscr{C} such

that

$$\pi_1 \circ h_1 = \pi_1 \circ h_2$$

$$\Downarrow$$

$$f \circ \pi_2 \circ h_1 = g \circ \pi_1 \circ h_1 = g \circ \pi_1 \circ h_2 = f \circ \pi_2 \circ h_2$$

Since f is monic by assumption, it follows that $\pi_2 \circ h_1 = \pi_2 \circ h_2$. As a result, the universal property of pullbacks implies that $h_1 = h_2$, as required.

Our next two results are quite useful and follow directly from the universal property of pullback (dually, pushout) squares.

Proposition 2.26. Let $f: X \to Y$ be a morphism in a category \mathscr{C} .

1. The commutative square

$$\begin{array}{ccc} X & & & X \\ \| & & & \downarrow^f \\ X & & & \downarrow^f \end{array} \\ X & & & & Y \end{array}$$

is a pullback if and only if f is a monomorphism.

2. The commutative square

$$\begin{array}{ccc} X & \stackrel{f}{\longrightarrow} Y \\ f & & & \\ Y & \stackrel{f}{\longrightarrow} & Y \end{array}$$

is a pushout if and only if f is an epimorphism.

Proposition 2.27 (Pasting law). Consider a commutative diagram of the form

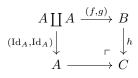
$$\begin{array}{cccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

in a category \mathcal{C} .

- 1. Suppose that the righthand square is a pullback. Then the total rectangle is a pullback if and only if the lefthand square is one.
- 2. Suppose that the lefthand square is a pushout. Then the total rectangle is a pushout if and only if the righthand square is one.

Corollary 2.28. The operations of forming pullbacks and forming pushouts are associative up to isomorphism.

All coequalizers $A \stackrel{f}{\underset{g}{\Rightarrow}} B \stackrel{h}{\longrightarrow} C$ can be obtained from taking binary coproducts and pushouts as follows.



Therefore, any category with binary coproducts and pushouts has coequalizers.

Moreover, any colimit of a sequence of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$
 (*)

is precisely the coequalizer of

$$\coprod_n X_n \xrightarrow[(u_{n+1} \circ f_n)_n]{\operatorname{Id}} \coprod_n X_n.$$

Therefore, any category with coequalizers and small coproducts has colimits of diagrams like (*). This fact can be generalized as follows.

Theorem 2.29 (Freyd).

- (i) If C has equalizers and small (resp. finite) products, then it has small (resp. finite) limits.
- (ii) If C has pullbacks and a terminal object, then it has finite limits.

Proof sketch.

1. Let $F: \mathscr{J} \to \mathscr{C}$ be any diagram with \mathscr{J} small. Consider the following two morphisms in \mathscr{C} :

$$f,g : \prod_{j \in ob \mathscr{J}} F_j \longrightarrow \prod_{\alpha: i \to j} F_j$$
$$\pi_{\alpha: i \to j} \circ f \equiv \pi_j$$
$$\pi_{\alpha: i \to j} \circ g \equiv F(\alpha) \circ \pi_i.$$

Then $\lim_{\mathscr{I}} F$ is precisely the equalizer of f and g.

Thanks to part (i), it suffices to show that C has equalizers and finite products. By assumption, there is some terminal object 1. Then any product A₁ × A₂ can be realized as the pullback of A₁ → 1 ← A₂. By induction, it follows that C has finite products. Moreover, for any morphisms f, g : A → B, note that any cone over the diagram

$$A \xrightarrow{(\mathrm{Id}_A,g)} A \times B \xleftarrow{(\mathrm{Id}_A,f)} A$$

yields morphisms $h : A \to C$ and $k : C \to A$ such that h = k and fk = gh. As a result, the pullback for this cospan is an equalizer of f and g, and thus our proof is complete.

We may view Example 2.19(1) as an instance of Theorem 2.29.

Next, let us show that adjoints interact nicely with (co)limits under mild conditions.

Proposition 2.30 (Left adjoints preserve colimits). Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors such that (F, G) is an adjoint pair. Let \mathcal{E} be small category. If $X : \mathcal{E} \to \mathcal{C}$ is a functor whose colimit exists, then

$$\operatorname{colim}_{\mathscr{E}}(F \circ X) \cong F\left(\operatorname{colim}_{\mathscr{E}}X\right)$$

Dually, if $Y:\mathscr{E}\to\mathscr{D}$ is a functor whose limit exists, then

$$\lim_{\mathscr{E}} (G \circ Y) \cong G\left(\lim_{\mathscr{E}} Y\right).$$

Proof. We have the following chain of natural bijections in $Y \in ob \mathscr{D}$:

$$\mathscr{D}\left(F\left(\operatorname{colim}_{\mathscr{E}}X\right),Y\right) \cong \mathscr{C}\left(\operatorname{colim}_{\mathscr{E}}X,G(Y)\right)$$
$$\cong \lim_{\mathscr{E}}\mathscr{C}(X(-),G(Y))$$
$$\cong \lim_{\mathscr{E}}\mathscr{D}(F(X(-)),Y)$$
$$\cong \operatorname{Fun}(\mathscr{E},\mathscr{D})(F \circ X,\Delta Y).$$

The second bijection exists because both sets can be identified with the components of all natural transformations $X \Rightarrow \Delta G(Y)$.

3 Fibers and Fibrations

Definition 3.1. Suppose \mathscr{C} has a terminal object 1. Let $f: X \to Y$ be a morphism in \mathscr{C} .

- 1. For any global element $p: 1 \to Y$ of Y, the fiber $f^{-1}(p)$ of f at p is the pullback of the cospan $1 \to Y \leftarrow X$.
- 2. The cofiber Y/X of f is the pushout of the span $1 \leftarrow X \rightarrow Y$.

For any functor $F : \mathscr{C} \to \mathscr{D}$, the *fiber category* $F^{-1}(Y)$ is the full subcategory of \mathscr{C} generated by those objects X such that F(X) = Y.

For each $Y \in ob \mathscr{D}$, there is a full and faithful functor $F^{-1}(Y) \to F/Y$ given by $X \mapsto (X, \mathrm{Id}_Y)$. We say that \mathscr{C} is a *precofibered category* over \mathscr{D} if F has a left adjoint given by

$$(Z, g: F(Z) \to Y) \mapsto g_*(Z)$$

Further, there is a full and faithful functor $F^{-1}(Y) \to Y/F$. We say that \mathscr{C} is a *prefibered category* over \mathscr{D} if this functor has a right adjoint given by $(Z, g: Y \to F(Z)) \mapsto g_*(Z)$.

Definition 3.2. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor.

1. Let $f: c' \to c$ be a morphism in \mathscr{C} . We say f is *cartesian* if for any morphism $f': c'' \to c$ in \mathscr{C} and any morphism $g: F(c'') \to F(c')$ in \mathscr{D} such that $Ff \circ g = Ff'$, there exists a unique morphism $\phi: c'' \to c$ such that $f' = f \circ \phi$ and $F\phi = g$.

In pictures,



2. We say that F is a fibration if for any $c \in ob \mathscr{C}$ and morphism $f : d \to Fc$ in \mathscr{D} , there is a cartesian morphism $\phi_f : c' \to c$ such that $F\phi_f = f$. Such a ϕ_f is called a *cartesian lifting* of f to c.

In this case, assuming the axiom of choice, we obtain a mapping $f \mapsto \phi_f$, known as a *cleavage* of F. If this respects the identity map and composition, then we call F a *normal* and *split* fibration, respectively.

Intuitively, if F is a fibration, then the fibers $F^{-1}(Y)$ depend functorially on $Y \in ob \mathscr{D}$.

Example 3.3.

- 1. Let the category **Mod** consist of pairs (R, M) as objects where R is a ring and M is a left R-module and pairs (f, \tilde{f}) as morphisms where $f : R \to R'$ is a ring homomorphism and $\tilde{f} : M \to M'$ is an R-linear map with M' viewed as an R-module via f. Then the forgetful functor $U : \mathbf{Mod} \to \mathbf{Ring}$ is a fibration.
- For any category C with pullbacks, consider the arrow category Ar(C) along with the codomain functor cod : Ar(C) → C defined by

$$\begin{array}{cccc} a \stackrel{f}{\longrightarrow} b & \mapsto & b \\ a \stackrel{\longrightarrow}{\longrightarrow} a' \\ \downarrow & \downarrow & \mapsto & b \rightarrow b'. \\ b \stackrel{\longrightarrow}{\longrightarrow} b' \end{array}$$

This is a fibration. Indeed, for any object $x \to y$ in $\operatorname{Ar}(\mathscr{C})$ and any morphism $z \to y$ in \mathscr{C} , the cartesian lifting of $z \to y$ to $x \to y$ is given by the pullback square