

Abstract

We introduce the concept of a universal property in category theory. The main sources for this talk are the following.

- *n*Lab.
- John Rognes’s *Lecture Notes on Algebraic K-Theory*, Ch. 4.
- Peter Johnstone’s lecture notes for “Category Theory” (Mathematical Tripos Part III, Michaelmas 2015), Ch. 4.
- Steve Awodey’s *Category Theory*, Ch. 5.6.

1 Universal arrows

Definition 1.1. Let \mathcal{C} be a category.

1. An object X of \mathcal{C} is *initial* if for each $Y \in \text{ob } \mathcal{C}$, there is a unique morphism $f : X \rightarrow Y$.
2. We say that X is *terminal* if for each $Z \in \text{ob } \mathcal{C}$, there is a unique morphism $g : Z \rightarrow X$.

Either condition is called a *universal property* of X .

Any property P of \mathcal{C} has a dual property P^{op} of \mathcal{C}^{op} obtained by interchanging the source and target of any arrow as well as the order of any composition in the sentence expressing P . Then P is true of \mathcal{C} iff P^{op} is true of \mathcal{C}^{op} .

Example 1.2. Being initial and being terminal are dual properties.

Lemma 1.3. *Any two initial objects of \mathcal{C} are unique up to unique isomorphism. The same holds for any two terminal objects of \mathcal{C} .*

Proof sketch. Let X and X' be two initial objects. Compose the two unique morphisms $X \rightarrow X'$ and $X' \rightarrow X$ to get an isomorphism between X and X' . Apply duality to this argument for the case of terminal objects. □

We can think of a universal property as follows. Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a functor and $X \in \text{ob } \mathcal{C}$. A *universal arrow from X to F* is an ordered pair (Y, f) with $Y \in \text{ob } \mathcal{D}$ and $f : X \rightarrow F(Y)$ a morphism of \mathcal{C} with the property that for any $X' \in \text{ob } \mathcal{D}$ and morphism $f' : X \rightarrow F(X')$ of \mathcal{C} , there exists a unique morphism $\hat{f} : Y \rightarrow X'$ of \mathcal{D} such that $F(\hat{f}) \circ f = f'$.

$$\begin{array}{ccc}
 X & \xrightarrow{f} & F(Y) \\
 & \searrow f' & \downarrow F(\hat{f}) \\
 & & F(X')
 \end{array}$$

Dually, a *universal arrow from F to X* is an ordered pair (Y, f) with $Y \in \text{ob } \mathcal{D}$ and $f : F(Y) \rightarrow X$ of \mathcal{C} with the property that for any $X' \in \text{ob } \mathcal{D}$ and morphism $f' : F(X') \rightarrow X$, there exists a unique morphism $\hat{f} : X' \rightarrow Y$ such that $f' = f \circ F(\hat{f})$.

$$\begin{array}{ccc} F(X') & \overset{F(\hat{f})}{\dashrightarrow} & F(Y) \\ & \searrow f' & \downarrow f \\ & & X \end{array}$$

To see why this notion of universality agrees with the original one, we first generalize the notion of an arrow category.

Definition 1.4.

1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $Y \in \text{ob } \mathcal{D}$. The *slice* or *left fiber category*, denoted by (F/Y) or $(F \downarrow Y)$, has as objects pairs (X, f) where $f : F(X) \rightarrow Y$ and as morphisms from $f : F(X) \rightarrow Y$ to $f' : F(X') \rightarrow Y$ morphisms $g : X \rightarrow X'$ such that $f = f' \circ F(g)$.
2. The *coslice* or *right fiber category*, denoted by (Y/F) or $(Y \downarrow F)$, has as objects pairs (X, f) where $f : Y \rightarrow F(X)$ and as morphisms from $f : Y \rightarrow F(X)$ to $f' : Y \rightarrow F(X')$ morphisms $g : X \rightarrow X'$ such that $f' = F(g) \circ f$.

If $F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ is opposite to the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and $Y \in \text{ob } \mathcal{D}$, then $(Y/F)^{\text{op}} = F^{\text{op}}/Y$. Thus, the left and right fiber categories are dual in the sense that $P(Y, F)$ is true of any right fiber category Y/F iff $P^{\text{op}}(Y, F)$ is true of any left fiber category F/Y .

Proposition 1.5. *Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a functor and $x \in \text{ob } \mathcal{C}$. Then $u : x \rightarrow Fr$ is a universal arrow from x to F iff it is an initial object of the coslice $(x \downarrow F)$. Dually, $u' : Fr' \rightarrow x$ is a universal arrow from F to x iff it is a terminal object of the same category.*

Proof. Suppose that u is universal and $f : x \rightarrow Fy$ is another object of $(x \downarrow F)$. Then there exists a unique $\hat{f} : r \rightarrow y$ such that $F(\hat{f}) \circ u = f$. Thus $F(\hat{f})$ is a unique morphism of the coslice.

Conversely, suppose that u is initial. Then for any object $f : x \rightarrow Fy$ of $(x \downarrow F)$, there exists a unique arrow $Sg : Fr \rightarrow Fy$ such that $Sg \circ u = f$. Hence setting $\hat{f} = g$ makes u a universal arrow. □

Corollary 1.6. *Any two universal arrows from x to F can be canonically identified by Lemma 1.3.*

2 (Co)limits

Definition 2.1. A *zero object* of \mathcal{C} is an object that is both initial and terminal.

Example 2.2. The unique initial object of **Set** is \emptyset , and the terminal objects are precisely the singleton sets. Hence there is no zero object. Moreover, there are no initial or terminal objects in $\text{iso}(\mathbf{Set})$.

Given $X \in \text{ob } \mathcal{C}$, the *undercategory* X/\mathcal{C} has as objects morphisms in \mathcal{C} of the form $i : X \rightarrow Y$ where X is fixed. Given $i : X \rightarrow Y$ and $i' : X \rightarrow Y'$ in $\text{ob } X/\mathcal{C}$, define the set of morphisms from i to i' as the morphisms $f : Y \rightarrow Y'$ where

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ & \searrow i' & \downarrow f \\ & & Y' \end{array}$$

commutes. (We call i the *structure morphism*.) Composition and identity carry over exactly from \mathcal{C} .

Likewise, given $x \in \text{ob } \mathcal{C}$, the *overcategory* \mathcal{C}/X has as objects morphisms in \mathcal{C} of the form $i : Y \rightarrow X$ where X is fixed. Given $i : Y \rightarrow X$ and $i' : Y' \rightarrow X$ in $\text{ob } \mathcal{C}/X$, define the set of morphisms from i to i' as the morphisms $f : Y \rightarrow Y'$ where

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ & \searrow i & \downarrow i' \\ & & X \end{array}$$

commutes. Composition and identity carry over exactly from \mathcal{C} .

Remark 2.3. If $X \in \text{ob } \mathcal{C}$, then $(X/\mathcal{C})^{\text{op}} = \mathcal{C}^{\text{op}}/X$. Thus, the under- and overcategories are dual in the sense that $P(X, \mathcal{C})$ is true of any undercategory X/\mathcal{C} iff $P^{\text{op}}(X, \mathcal{C})$ is true of any overcategory \mathcal{C}/X .

Lemma 2.4. *For any $X \in \mathcal{C}$, the identity morphism on X is an initial object X/\mathcal{C} . Dually, it is a terminal object in \mathcal{C}/X .*

Proof. Any morphism $i : X \rightarrow Y$ is itself the unique morphism from Id_X to i . □

Lemma 2.5. *Let X be an initial object of \mathcal{C} . The identity morphism on X is a zero object \mathcal{C}/X . Dually, if $Y \in \text{ob } \mathcal{C}$ is terminal, then Id_Y is a zero object in Y/\mathcal{C} .*

Proof. We already know that Id_X is terminal. If $p : Y \rightarrow X$ is an object in \mathcal{C}/X , then there is a unique morphism $f : X \rightarrow Y$. Then $f \circ p$ must equal Id_X . □

Example 2.6. Let (X, x) be a pointed set with $X = \{x\}$. Let \mathbf{Set}_* denotes the category of pointed sets with base point preserving functions. Since $\mathbf{Set}_* \cong X/\mathbf{Set}$, it follows that X is a zero object in \mathbf{Set}_* .

Given a morphism $\alpha : X \rightarrow Z$ in \mathcal{C} , define the *under-and-overcategory* $(X/\mathcal{C}/Z)_\alpha$ as having triples (Y, i, p) as objects where $i : X \rightarrow Y$ and $p : Y \rightarrow Z$ are morphisms in \mathcal{C} such that $p \circ i = \alpha$. Define the set of morphisms from (Y, i, p) to (Y', i', p') as the set of morphisms $f : Y \rightarrow Y'$ such that

$$\begin{array}{ccc} X & \xrightarrow{i'} & Y' \\ i \downarrow & \begin{array}{c} \text{blue } \nearrow \\ \text{red } \searrow \end{array} & \downarrow p' \\ Y & \xrightarrow{p} & Z \end{array}$$

commutes. If $\alpha = \text{Id}_X$, then we call $(X/\mathcal{C}/X)_{\text{Id}_X}$ the category of *retractive* objects over X , with each triple (Y, i, p) being a retraction of Y onto X .

Example 2.7. If $F : \mathcal{C} \rightarrow \mathcal{C}$ is the identity functor, then the undercategory Y/\mathcal{C} equals the right fiber category Y/F , and the overcategory \mathcal{C}/Y equals the left fiber category F/Y .

Let \mathcal{J} be a category. A *diagram of shape \mathcal{J} in \mathcal{C}* is a functor $F : \mathcal{J} \rightarrow \mathcal{C}$.

Definition 2.8. Given a functor $F : \mathcal{J} \rightarrow \mathcal{C}$ and $X \in \text{ob } \mathcal{C}$, a *cone over F* consists of an *apex* $X \in \text{ob } \mathcal{C}$ and *legs* $f_j : X \rightarrow F(j)$ for each $j \in \text{ob } \mathcal{J}$ such that for any morphism $\alpha : j \rightarrow j'$, the triangle

$$\begin{array}{ccc} X & \xrightarrow{f_j} & F(j) \\ & \searrow f_{j'} & \downarrow F\alpha \\ & & F(j') \end{array}$$

commutes.

This is simply a natural transformation $\Delta_{\mathcal{J}} X \Rightarrow F$ where $\Delta_{\mathcal{J}} X$ denotes the constant functor on \mathcal{J} at X . If \mathcal{J} is small, then $\Delta_{\mathcal{J}}$ is a functor from \mathcal{C} to $\mathbf{Fun}(\mathcal{J}, \mathcal{C})$.

Definition 2.9.

1. The *category of cones over F* is the right fiber category X/F .
2. The *category of cocones under F* is the left fiber category F/X .

Definition 2.10 (Colimit). Let \mathcal{C} and \mathcal{D} be categories and $g : Y \rightarrow Z$ a morphism in \mathcal{D} . Let $\Delta_{\mathcal{C}} g : \Delta_{\mathcal{C}} Y \Rightarrow \Delta_{\mathcal{C}} Z$ be the natural transformation with components $X \mapsto g$.

1. A *colimit* $\text{colim}_{\mathcal{C}} F$ of the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an object Y of \mathcal{D} together with a natural transformation $i : F \Rightarrow \Delta_{\mathcal{C}} Y$ such that for any $Z \in \text{ob } \mathcal{D}$ and any natural transformation $j : F \Rightarrow \Delta_{\mathcal{C}} Z$, there is a unique morphism $g : Y \rightarrow Z$ such that $j = \Delta_{\mathcal{C}} g \circ i$.
2. We say that \mathcal{D} *admits/has \mathcal{C} -shaped colimits* if each functor $G : \mathcal{C} \rightarrow \mathcal{D}$ has a colimit.
3. We say that \mathcal{D} is *cocomplete* if each functor $G : \mathcal{C} \rightarrow \mathcal{D}$ with \mathcal{C} small has a colimit.

If \mathcal{C} is small, then a colimit of $F : \mathcal{C} \rightarrow \mathcal{D}$ is just an initial object in the right fiber category $F/\Delta_{\mathcal{C}}$, which has as objects pairs $(Z, j : F \rightarrow \Delta Z)$ and as morphisms from (Y, i) to (Z, j) the morphisms $g : Y \rightarrow Z$ in \mathcal{D} such that $\Delta g \circ i = j$.

Example 2.11. If \mathcal{C} is the empty category, then the empty functor $F : \mathcal{C} \rightarrow \mathcal{D}$ satisfies $F/\Delta_{\mathcal{C}} \cong \mathcal{D}$, so that the colimit of F is exactly the initial object of \mathcal{D} .

Proposition 2.12. *There is a natural bijection $\mathcal{D}(Y, Z) \cong \mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, \Delta Z)$ if and only if $Y = \text{colim}_{\mathcal{C}} F$.*

Lemma 2.13. *Any two colimits are unique up to unique isomorphism.*

Proof. When \mathcal{C} is small, this follows immediately from Lemma 1.3. Notice, however, that our proof of Lemma 1.3 does *not* require that \mathcal{C} be locally small (a property which Rognes stipulates of any category). \square

Remark 2.14. Assume that \mathcal{D} has \mathcal{C} -shaped colimits and that \mathcal{C} is small. Then a (possibly global) choice function $\text{colim}_{\mathcal{C}} : \mathbf{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D}$ given by choosing a colimit for each functor determines a functor that is left adjoint to the constant diagram functor $\Delta_{\mathcal{C}} : \mathcal{D} \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{D})$. Indeed, for any functor $F : \mathcal{C} \rightarrow \mathcal{D}$, there is a bijection $\mathcal{D}(\text{colim}_{\mathcal{C}} F, Z) \cong \mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, \Delta_{\mathcal{C}} Z)$.

Definition 2.15 (Limit). A *limit* of the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a colimit of $F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$.

Explicitly, a limit for $F : \mathcal{C} \rightarrow \mathcal{D}$ is an object Z of \mathcal{D} along with a natural transformation $p : \Delta_{\mathcal{C}} Z \Rightarrow F$ such that for any $Y \in \text{ob } \mathcal{D}$ and any natural transformation $q : \Delta_{\mathcal{C}} Y \Rightarrow F$, there is a unique morphism $g : Y \rightarrow Z$ such that $q = p \circ \Delta_{\mathcal{C}} g$.

Note that the colimit of a functor F is exactly the limit of F^{op} . Hence *limit* and *colimit* are dual properties, and our results so far for colimits can be dualized for limits.

Definition 2.16 ((Co)product). Let \mathcal{J} be a discrete small category. Consider a diagram $\{A_i\}_{i \in \text{ob } \mathcal{J}}$ of shape \mathcal{J} .

1. The limit of $\{A_i\}_i$ is called the *product* $\prod_i A_i$, equipped with projections $\pi_i : \prod_i A_i \rightarrow A_i$ such that for every $f_i : U \rightarrow A_i$ there exists a unique map $f := (f_i) : U \rightarrow \prod_i A_i$ satisfying $\pi_i \circ f = f_i$.
2. The colimit of $\{A_i\}_i$ is called the *coproduct* $\coprod_i A_i$, equipped with inclusions $u_i : A_i \rightarrow \coprod_i A_i$ such that for any $f_i : A_i \rightarrow Y$, there exists a unique map $f := (f_i) : \coprod_i A_i \rightarrow Y$ satisfying $f_i = f \circ u_i$.

Familiar examples of limits include cartesian products and direct products, whereas familiar examples of colimits include disjoint unions and free products.

Example 2.17.

- (1) Consider any small diagram $F : \mathcal{J} \rightarrow \mathbf{Set}$. On the one hand,

$$\text{colim}_j F_j \cong \left(\coprod_{j \in \text{ob } \mathcal{J}} F_j \right) / \sim$$

where \sim is the smallest equivalence relation such that $F_j \ni f_j \sim f_{j'} \in F_{j'}$, whenever $F(\psi)(f_j) = f_{j'}$ for some $\psi : j \rightarrow j'$.

On the other hand,

$$\lim_j F_j \cong \left\{ (f_j)_j \in \prod_{j \in \text{ob } \mathcal{J}} F_j \mid \forall \psi : j \rightarrow j' \text{ in } \mathcal{J}, F(\psi)(f_j) = f_{j'} \right\}.$$

We have shown that \mathbf{Set} is both complete and cocomplete.

- (2) Let A be any set. Define the *cumulative hierarchy* $V_n(A)$ of rank $n < \omega$ over A along with a countable sequence

$$V_0 \xrightarrow{v_0} V_1 \xrightarrow{v_1} V_2 \longrightarrow \cdots \longrightarrow V_n \xrightarrow{v_n} V_{n+1} \longrightarrow \cdots$$

of maps recursively by

$$\begin{aligned} V_0(A) &= A \\ V_{n+1}(A) &= A \coprod \mathcal{P}(V_n(A)) \\ v_0 : A &\hookrightarrow A \coprod \mathcal{P}(A), & a &\mapsto a \\ v_{n+1} : A \coprod \mathcal{P}(V_n(A)) &\rightarrow A \coprod \mathcal{P}(V_{n+1}(A)), & (\text{Id}_A, \mathcal{P}(V_n(A))) &. \end{aligned}$$

Let $V_\omega(A) = \text{colim}_{n < \omega} V_n(A)$, which exists by part (1). Then $V_\omega(\emptyset)$ is exactly the set of all hereditarily finite sets. To see that $V_\omega(-)$ is a functor $\mathbf{Set} \rightarrow \mathbf{Set}$, let $f : A \rightarrow B$ be a function. Then we can build a cocone

$$\begin{array}{ccccccc} V_0(A) & \xrightarrow{v_0} & V_1(A) & \longrightarrow & \cdots & \longrightarrow & V_n(A) & \xrightarrow{v_n} & V_{n+1}(A) \\ f_0 \equiv f \downarrow & & \downarrow (f, \mathcal{P}(f)) & & & & \downarrow (f, \mathcal{P}(f_{n-1})) & & \downarrow (f, \mathcal{P}(f_n)) \\ V_0(B) & \longrightarrow & V_1(B) & \longrightarrow & \cdots & \longrightarrow & V_n(B) & \longrightarrow & V_{n+1}(B) & \longrightarrow & V_\omega(B) \end{array}$$

under $\{V_n(A)\}_n$ recursively. By the universal property of colimits, there exists a unique map $V_\omega(A) \rightarrow V_\omega(B)$, so that $V_\omega(-)$ is functorial.

Let \mathcal{J} be a category of the form $\bullet \rightrightarrows \bullet$. Then a diagram D of shape \mathcal{J} looks like $A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B$. A cone over D with apex C and legs $f_1 : C \rightarrow A$ and $f_2 : C \rightarrow B$ satisfies $f \circ f_1 = f_2 \circ g$.

Definition 2.18 ((Co)equalizer).

1. If such an object C together with f_1 is the limit of D , then we say it is the *equalizer* of f and g .
2. The colimit of D is called the *coequalizer* of f and g .

Example 2.19. The equalizer in **Set** of $f, g : X \rightarrow Y$ is the subset $X' := \{x \in X : f(x) = g(x)\}$ together with the inclusion function $X' \hookrightarrow X$.

The coequalizer of (f, g) is precisely Y/\sim together with the quotient map on B where \sim is the smallest equivalence relation under which $f(x) \sim g(x)$ for every x .

It is easy to check that any equalizer $f : C \rightarrow A$ must be monic. Further, if f is split epic, i.e., has a section $g : A \rightarrow C$, as well, then f is an isomorphism. For, in this case, $f \circ (g \circ f) = \text{Id}_A \circ f = f \circ \text{Id}_C$. As f is monic, we have that $g \circ f = \text{Id}_C$, so that g is an inverse of f .

Next, let \mathcal{J} be a category of the form $\bullet \rightarrow \bullet \leftarrow \bullet$. Then a diagram of this shape looks like $B \xrightarrow{f} C \xleftarrow{g} A$, and a cone over this diagram looks like

$$\begin{array}{ccc} E & \xrightarrow{j} & A \\ \downarrow i & \searrow \alpha & \downarrow g \\ B & \xrightarrow{f} & C \end{array}$$

Definition 2.20 (Pullback). If such an object E together with i and j is the limit of this diagram, then we call it the *pullback* of f and g , denoted by $B \times_C A$.

The universal property of a pullback square states that for any commutative diagram of the form

$$\begin{array}{ccc} & & A \\ & \searrow & \downarrow f \\ Z & \xrightarrow{\quad} & B \xrightarrow{g} C \\ & \swarrow & \downarrow \pi_B \\ & & B \times_C A \xrightarrow{\pi_A} A \end{array}$$

there is a unique *mediating map* $Z \rightarrow B \times_C A$ fitting into it.

If we perform a dual construction for \mathcal{J}^{op} , then the colimit of the resulting diagram is called the *pushout*, denoted by $B \cup_C A$. The universal property of a pullback square states that for any commutative diagram of the form

$$\begin{array}{ccc} B \times_C A & \xrightarrow{\pi_A} & A \\ \downarrow \pi_B & & \downarrow f \\ B & \xrightarrow{g} & C \end{array} \quad , \quad \begin{array}{ccc} & & Z \\ & \swarrow & \downarrow \\ & & B \times_C A \end{array}$$

there is a unique mediating map $B \cup_C A \rightarrow Z$ fitting into it.

Example 2.21.

1. The pullback in **Set** of $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ is precisely $\{(x, y) \in X \times Y : f(x) = g(y)\}$, called the *fibred product* of X and Y over Z .
2. The pushout in **Set** of $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ is precisely the quotient of $X \amalg Y$ by the equivalence relation \sim generated by the formula $(\forall z \in Z) (f(z) \sim g(z))$. We call $X \amalg Y / \sim$ the *fibred sum* of X and Y under Z .

Example 2.22. Let $\mathbf{FinSet}_{\text{mono}}$ denote the category of all finite sets with injective functions as arrows. The category of *nominal sets* consists of all pullback-preserving functors $\mathbf{FinSet}_{\text{mono}} \rightarrow \mathbf{Set}$ with natural transformations as arrows. These correctly encode the syntax of functional programming languages modulo renaming of bound variables (which is necessary for substitution).

Proposition 2.23. *The pullback of a monomorphism in a category \mathcal{C} is again a monomorphism in \mathcal{C} .*

Proof. Consider any pullback square

$$\begin{array}{ccc} B \times_C A & \xrightarrow{\pi_2} & A \\ \pi_1 \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

in \mathcal{C} where f is monic. We must show that π_1 is monic. Let $h_1, h_2 : B' \rightarrow B \times_C A$ be morphisms in \mathcal{C} such that

$$\begin{aligned} \pi_1 \circ h_1 &= \pi_1 \circ h_2 \\ \Downarrow \\ f \circ \pi_2 \circ h_1 &= g \circ \pi_1 \circ h_1 = g \circ \pi_1 \circ h_2 = f \circ \pi_2 \circ h_2. \end{aligned}$$

Since f is monic by assumption, it follows that $\pi_2 \circ h_1 = \pi_2 \circ h_2$. As a result, the universal property of pullbacks implies that $h_1 = h_2$, as required. \square

Our next two results are quite useful and follow directly from the universal property of pullback (dually, pushout) squares.

Proposition 2.24. *Let $f : X \rightarrow Y$ be a morphism in a category \mathcal{C} .*

1. *The commutative square*

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \parallel & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback if and only if f is a monomorphism.

2. *The commutative square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ f \downarrow & & \parallel \\ Y & \xlongequal{\quad} & Y \end{array}$$

is a pushout if and only if f is an epimorphism.

Proposition 2.25 (Pasting law). Consider a commutative diagram of the form

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

in a category \mathcal{C} .

1. Suppose that the righthand square is a pullback. Then the total rectangle is a pullback if and only if the lefthand square is one.

2. Suppose that the lefthand square is a pushout. Then the total rectangle is a pushout if and only if the righthand square is one.

Corollary 2.26. The operations of forming pullbacks and forming pushouts are associative up to isomorphism.

All coequalizers $A \begin{smallmatrix} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{g} \end{smallmatrix} B \xrightarrow{h} C$ can be obtained from taking binary coproducts and pushouts as follows.

$$\begin{array}{ccc} A \amalg A & \xrightarrow{(f,g)} & B \\ (\text{Id}_A, \text{Id}_A) \downarrow & \lrcorner & \downarrow h \\ A & \longrightarrow & C \end{array}$$

Therefore, any category with binary coproducts and pushouts has coequalizers.

Moreover, any colimit of a sequence of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \quad (*)$$

is precisely the coequalizer of

$$\prod_n X_n \begin{array}{c} \xrightarrow{\text{Id}} \\ \xrightarrow{(u_{n+1} \circ f_n)_n} \end{array} \prod_n X_n.$$

Therefore, any category with coequalizers and small coproducts has colimits of diagrams like $(*)$. This fact can be generalized as follows.

Theorem 2.27 (Freyd).

(i) If \mathcal{C} has equalizers and small (resp. finite) products, then it has small (resp. finite) limits.

(ii) If \mathcal{C} has pullbacks and a terminal object, then it has finite limits.

Proof sketch.

1. Let $F : \mathcal{J} \rightarrow \mathcal{C}$ be any diagram with \mathcal{J} small. Consider the following two morphisms in \mathcal{C} :

$$\begin{aligned} f, g : \prod_{j \in \text{ob } \mathcal{J}} F_j &\rightarrow \prod_{\alpha: i \rightarrow j} F_j \\ \pi_{\alpha: i \rightarrow j} \circ f &\equiv \pi_j \\ \pi_{\alpha: i \rightarrow j} \circ g &\equiv F(\alpha) \circ \pi_i. \end{aligned}$$

Then $\lim_{\mathcal{J}} F$ is precisely the equalizer of f and g .

2. Thanks to part (i), it suffices to show that \mathcal{C} has equalizers and finite products. By assumption, there is some terminal object 1 . Then any product $A_1 \times A_2$ can be realized as the pullback of $A_1 \rightarrow 1 \leftarrow A_2$. By induction, it follows that \mathcal{C} has finite products. Moreover, for any morphisms $f, g : A \rightarrow B$, note that any cone over the diagram

$$A \xrightarrow{(\text{Id}_A, g)} A \times B \xleftarrow{(\text{Id}_A, f)} A$$

yields morphisms $h : A \rightarrow C$ and $k : C \rightarrow A$ such that $h = k$ and $fk = gh$. As a result, the pullback for this cospan is an equalizer of f and g , and thus our proof is complete. □

We may view Example 2.17(1) as an instance of Theorem 2.27.

Next, let us show that adjoints interact nicely with (co)limits under mild conditions.

Proposition 2.28 (Left adjoints preserve colimits). *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors such that (F, G) is an adjoint pair. Let \mathcal{E} be small category. If $X : \mathcal{E} \rightarrow \mathcal{C}$ is a functor whose colimit exists, then*

$$\text{colim}_{\mathcal{E}}(F \circ X) \cong F \left(\text{colim}_{\mathcal{E}} X \right).$$

Dually, if $Y : \mathcal{E} \rightarrow \mathcal{D}$ is a functor whose limit exists, then

$$\lim_{\mathcal{E}}(G \circ Y) \cong G \left(\lim_{\mathcal{E}} Y \right).$$

Proof. We have the following chain of natural bijections in $Y \in \text{ob } \mathcal{D}$:

$$\begin{aligned} \mathcal{D} \left(F \left(\text{colim}_{\mathcal{E}} X \right), Y \right) &\cong \mathcal{C} \left(\text{colim}_{\mathcal{E}} X, G(Y) \right) \\ &\cong \lim_{\mathcal{E}} \mathcal{C}(X(-), G(Y)) \\ &\cong \lim_{\mathcal{E}} \mathcal{D}(F(X(-)), Y) \\ &\cong \mathbf{Fun}(\mathcal{E}, \mathcal{D})(F \circ X, \Delta Y). \end{aligned}$$

The second bijection exists because both sets can be identified with the components of all natural transformations $X \Rightarrow \Delta G(Y)$. □

3 Fibers and Fibrations

Definition 3.1. Suppose \mathcal{C} has a terminal object 1 . Let $f : X \rightarrow Y$ be a morphism in \mathcal{C} .

1. Given a global element $p : 1 \rightarrow Y$ of Y , the *fiber* $f^{-1}(p)$ of f at p is the pullback of the cospan $1 \rightarrow Y \leftarrow X$.
2. The *cofiber* Y/X of f is the pushout of the span $1 \leftarrow X \rightarrow Y$.

For any functor $F : \mathcal{C} \rightarrow \mathcal{D}$, the *fiber category* $F^{-1}(Y)$ is the full subcategory of \mathcal{C} generated by those objects X such that $F(X) = Y$.

For each $Y \in \text{ob } \mathcal{D}$, there is a full and faithful functor $F^{-1}(Y) \rightarrow F/Y$ given by $X \mapsto (X, \text{Id}_Y)$. We say that \mathcal{C} is a *precofibered category* over \mathcal{D} if F has a left adjoint given by

$$(Z, g : F(Z) \rightarrow Y) \mapsto g_*(Z).$$

Further, there is a full and faithful functor $F^{-1}(Y) \rightarrow Y/F$. We say that \mathcal{C} is a *prefibered category* over \mathcal{D} if this functor has a right adjoint given by $(Z, g : Y \rightarrow F(Z)) \mapsto g_*(Z)$.

Definition 3.2. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

1. Let $f : c' \rightarrow c$ be a morphism in \mathcal{C} . We say f is *cartesian* if for any morphism $f' : c'' \rightarrow c$ in \mathcal{C} and any morphism $g : F(c'') \rightarrow F(c')$ in \mathcal{D} such that $Ff \circ g = Ff'$, there exists a unique morphism $\phi : c'' \rightarrow c'$ such that $f' = f \circ \phi$ and $F\phi = g$.

In pictures,

$$\begin{array}{ccc} F(c'') & \overset{g}{\dashrightarrow} & F(c') \\ & \searrow Ff' & \downarrow Ff \\ & & F(c) \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} c'' & \overset{\exists! \phi}{\dashrightarrow} & c' \\ & \searrow f' & \downarrow f \\ & & c \end{array}, \quad \phi \xrightarrow{F} g.$$

2. We say that F is a *fibration* if for any $c \in \text{ob } \mathcal{C}$ and morphism $f : d \rightarrow Fc$ in \mathcal{D} , there is a cartesian morphism $\phi_f : c' \rightarrow c$ such that $F\phi_f = f$. Such a ϕ_f is called a *cartesian lifting* of f to c .

In this case, assuming the axiom of choice, we obtain a mapping $f \mapsto \phi_f$, known as a *cleavage* of F . If this respects the identity map and composition, then we call F a *normal* and *split* fibration, respectively.

Intuitively, if F is a fibration, then the fibers $F^{-1}(Y)$ depend functorially on $Y \in \text{ob } \mathcal{D}$.

Example 3.3.

1. Let the category **Mod** consist of pairs (R, M) as objects where R is a ring and M is a left R -module and pairs (f, \tilde{f}) as morphisms where $f : R \rightarrow R'$ is a ring homomorphism and $\tilde{f} : M \rightarrow M'$ is an R -linear map with M' viewed as an R -module via f . Then the forgetful functor $U : \mathbf{Mod} \rightarrow \mathbf{Ring}$ is a fibration.
2. For any category \mathcal{C} with pullbacks, consider the arrow category $\text{Ar}(\mathcal{C})$ along with the codomain functor $\text{cod} : \text{Ar}(\mathcal{C}) \rightarrow \mathcal{C}$ defined by

$$\begin{array}{ccc} a \xrightarrow{f} b & \mapsto & b \\ \begin{array}{ccc} a & \longrightarrow & a' \\ \downarrow & & \downarrow \\ b & \longrightarrow & b' \end{array} & \mapsto & b \rightarrow b'. \end{array}$$

This is a fibration. Indeed, for any object $x \rightarrow y$ in $\text{Ar}(\mathcal{C})$ and any morphism $z \rightarrow y$ in \mathcal{C} , the cartesian lifting of $z \rightarrow y$ to $x \rightarrow y$ is given by the pullback square

$$\begin{array}{ccc} z \times_y x & \longrightarrow & x \\ \downarrow & & \downarrow \\ z & \longrightarrow & y \end{array}.$$