Perry Hart
K-theory reading seminar
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Abstract

We continue looking at higher Waldhausen K-theory by presenting several of its key theorems. At the end, we see an encoding of Waldhausen K-theory as the infinite loop space of a sort of spectrum. The main sources for this talk are the following.

- nLab
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Sect. V.2
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 8

1 Extension and additivity

Let $\mathscr{B}$ and $\mathscr{C}$ be Waldhausen categories. We say that $F' \rightarrowtail F \twoheadrightarrow F''$ is a *short exact sequence* or *cofiber sequence of exact functors* $\mathscr{B} \to \mathscr{C}$ if

(i) the sequence $F'(B) \rightarrowtail F(B) \twoheadrightarrow F''(B)$ is a cofiber sequence for every $B \in \operatorname{ob} \mathscr{B}$ and

(ii) the map $F(A) \cup_{F'(A)} F'(B) \rightarrowtail F(B)$ is a cofibration in $\mathscr{C}$ for every $A \rightarrowtail B$ in $\mathscr{B}$.

Let $\eta : \; A \rightarrowtail B \twoheadrightarrow C$ be an object in $S_2\mathscr{C}$. Define the source s, target t, and quotient q functors $S_2\mathscr{C} \to \mathscr{C}$ by $s(\eta) = A$, $t(\eta) = B$, and $q(\eta) = C$, respectively. Then $s \rightarrowtail t \twoheadrightarrow q$ is a cofiber sequence of functors.

Since defining a cofiber sequence of exact functors $\mathscr{B} \to \mathscr{C}$ is equivalent to defining an exact functor $\mathscr{B} \to S_2\mathscr{C}$, we may restrict our attention to $s \rightarrowtail t \twoheadrightarrow q$ when proving assertions about a given cofiber sequence of exact functors $\mathscr{B} \to \mathscr{C}$. (We say that $S_2\mathscr{C}$ is *universal* in this sense.)

Theorem 1.1 (Extension). *The exact functor $(s, q) : S_2\mathscr{C} \to \mathscr{C} \times \mathscr{C}$ induces a homotopy equivalence $K(S_2\mathscr{C}) \simeq K(\mathscr{C}) \times K(\mathscr{C})$. The functor $\coprod : (A, B) \to (A \rightarrowtail A \coprod B \twoheadrightarrow B)$ is a homotopy inverse.*

Proof sketch. Let $\mathscr{C}^w_m$ denote the category of m-length sequences of weak equivalences. For each n, define $s_n\mathscr{C}^w_m$ as the commutative diagram

$$\begin{array}{ccccccc}
X_1^0 & \rightarrowtail & X_2^0 & \rightarrowtail & \cdots & \rightarrowtail & X_n^0 \\
\sim\downarrow & & \sim\downarrow & & & & \sim\downarrow \\
X_1^1 & \rightarrowtail & X_2^1 & \rightarrowtail & \cdots & \rightarrowtail & X_n^1 \\
\sim\downarrow & & \sim\downarrow & & & & \sim\downarrow \\
\vdots & & \vdots & & & & \vdots \\
\sim\downarrow & & \sim\downarrow & & & & \sim\downarrow \\
X_1^m & \rightarrowtail & X_2^m & \rightarrowtail & \cdots & \rightarrowtail & X_n^m
\end{array} .$$

This is naturally isomorphic to an (m,n)-bisimplex in $N_\bullet wS_\bullet\mathscr{C}$, which is thus isomorphic to the bisimplicial set $s_\bullet\mathscr{C}^w_{(-)}$. One can show that the source s and quotient q functors $S_2\mathscr{C}\to\mathscr{C}$ induce a homotopy equivalence $s\times q : s_\bullet S_2(\mathscr{C}^w_m)\to s_\bullet\mathscr{C}^w_m\times s_\bullet\mathscr{C}^w_m$ for each m. Thus, we get a homotopy equivalence

$$s_\bullet S_2(\mathscr{C}^w_{(-)})\simeq s_\bullet\mathscr{C}^w_{(-)}\times s_\bullet\mathscr{C}^w_{(-)}$$

between bisimplicial sets. But we already have that $s_\bullet\mathscr{C}^w_{(-)}\cong N_\bullet wS_\bullet\mathscr{C}$, thereby completing our proof. □

Recall that $|wS_\bullet\mathscr{C}|$ is an H-space via the map

$$\coprod : |wS_\bullet\mathscr{C}|\times|wS_\bullet\mathscr{C}|\cong|wS_\bullet\mathscr{C}\times wS_\bullet\mathscr{C}|\to|wS_\bullet\mathscr{C}|\,. \tag{$\star$}$$

This produces an H-space structure $(K(\mathscr{C}),+)$.

Theorem 1.2 (Additivity). *Let $F'\rightarrowtail F\twoheadrightarrow F''$ be a short exact sequence of exact functors $\mathscr{B}\to\mathscr{C}$. Then $F_*\simeq F'_*+F''_*$ as maps $K(\mathscr{B})\to K(\mathscr{C})$, so that*

$$F_*=F'_*+F''_*$$

as maps $K_i(\mathscr{B})\to K_i(\mathscr{C})$.

Proof. As $S_2\mathscr{C}$ is universal, it suffices to prove that $t_*\simeq s_*+q_*$. Notice that the two composites

$$\mathscr{C}\times\mathscr{C}\xrightarrow{\coprod}S_2\mathscr{C}\underset{s\coprod q}{\overset{t}{\rightrightarrows}}\mathscr{C}$$

are the same. Theorem 1.1 implies that $K\left(\coprod\right):K(\mathscr{C})\times K(\mathscr{C})\to K(S_2\mathscr{C})$ is a homotopy equivalence. Since the H-space structure on $K(\mathscr{C})$ is induced by $\coprod$, we conclude that $t_*\simeq s_*+q_*$. □

Definition 1.3. We say that a sequence

$$*\to A_n\to\cdots\to A_0\to *$$

is *admissibly exact* if each morphism in the sequences can be written as a cofiber sequence

$$A_{i+1}\twoheadrightarrow B_i\rightarrowtail A_i.$$

Corollary 1.4. *Suppose that*

$$*\to F^0\to F^1\to\cdots\to F^n\to *$$

is an admissibly exact sequence of exact functors $\mathscr{B}\to\mathscr{C}$. Then we have an equality

$$\sum_i(-1)^iF^i_*=0$$

of maps $K_i(\mathscr{B})\to K_i(\mathscr{C})$.

Corollary 1.5. *Let $F' \rightarrowtail F \twoheadrightarrow F''$ be a short exact sequence of exact functors $\mathscr{B} \to \mathscr{C}$. Then*

$$F''_* \simeq F_* - F_* \simeq 0.$$

This implies that the homotopy fiber of $F''_* : K(\mathscr{B}) \to K(\mathscr{C})$ is homotopy equivalent to $K(\mathscr{B}) \vee \Omega K(\mathscr{C})$.

Let $\mathscr{C}$ be a Waldhausen category. Recall the arrow category $\mathrm{Ar}(\mathscr{C})$ of $\mathscr{C}$ consisting of morphisms in $\mathscr{C}$ as objects and commutative squares as morphisms. Let s and t denote the source and target functors $\mathrm{Ar}(\mathscr{C}) \to \mathscr{C}$, respectively.

Definition 1.6. A functor $T : \mathrm{Ar}(\mathscr{C}) \to \mathscr{C}$ is a *(mapping) cylinder functor* on $\mathscr{C}$ if it comes equipped with natrual transformations $j_1 : s \Rightarrow T$, $j_2 : t \Rightarrow T$, and $p : T \Rightarrow t$ such that for any $f : A \to B$, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] & B \arrow[l, "j_2"'] \arrow[dl, "="] \\ & B & \end{tikzcd}.$$

Moreover, T must satisfy the following axioms.

(1) T sends every initial morphism $* \to A$ to A for any $A \in \mathrm{ob}\,\mathscr{C}$.

(2) The map $j_1 \coprod j_2 : A \coprod B \rightarrowtail T(f)$ is a cofibration for any $f : A \to B$.

(3) Given a morphism $(a, b) : f \to f'$ in $\mathrm{Ar}(\mathscr{C})$, if both a and b are weak equivalences in $\mathscr{C}$, then so is $T(f) \to T(f')$.

(4) Given a morphism $(a, b) : f \to f'$ in $\mathrm{Ar}(\mathscr{C})$, if both a and b are cofibrations in $\mathscr{C}$, then so is $T(f) \to T(f')$. Also, the map

$$A' \coprod_A T(f) \coprod_B B' \to T(f')$$

induced by axiom (2) is a cofibration in $\mathscr{C}$.

(5) (*Cylinder axiom*) The map $p : T(f) \to B$ is a weak equivalence in $\mathscr{C}$.

Terminology. Let T be a cylinder functor on $\mathscr{C}$.

1. We call $T(A \to *)$ the *cone* of A, denoted by $\mathrm{cone}(A)$.

2. We call $\mathrm{cone}(A)/A$ the *suspension* of A, denoted by ΣA.

Corollary 1.7. *The induced suspension map $\Sigma : K(\mathscr{C}) \to K(\mathscr{C})$ is a homotopy inverse for the H-space structure* ($\star$).

Proof. Note that axiom (3) gives us a cofiber sequence $A \rightarrowtail \mathrm{cone}(A) \twoheadrightarrow \Sigma A$. Therefore, $1 \rightarrowtail \mathrm{cone} \twoheadrightarrow \Sigma$ is an exact sequence of functors. By the cylinder axiom, we know that cone is null-homotopic. It follows by Theorem 1.2 that $\Sigma_* + 1 = \mathrm{cone}_* = *$. $\square$

2 Localization

Let $\mathscr{C}$ be a category with cofibrations. Equip it with two Waldhausen subcategories $v(\mathscr{C})$ and $w(\mathscr{C})$ of weak equivalences such that $v(\mathscr{C}) \subset w(\mathscr{C})$. Assume that $(\mathscr{C}, w)$ admits a cylinder functor. Suppose that $w(\mathscr{C})$ is saturated and closed under extensions.

Let $\mathscr{C}^w$ denote the Waldhausen subcategory of $(\mathscr{C}, v)$ consisting of all A such that $* \to A$ belongs to $w(\mathscr{C})$.

Are the initial morphisms the only weak equivalences?

Theorem 2.1 (Waldhausen localization). *The sequence*

$$K(\mathscr{C}^w) \to K(\mathscr{C}, v) \to K(\mathscr{C}, w)$$

is a homotopy fibration sequence.

Proof sketch. Recall that a small bicategory is a bisimplicial set such that each row/column is the nerve of a category. Note that $v_{(-)}w_{(-)}\mathscr{C}$ is a bicategory whose bimorphisms are commutative squares of the form

$$\begin{tikzcd} (-) \arrow[r, "w'"] \arrow[d, "v"'] & (-) \arrow[d, "v'"] \\ (-) \arrow[r, "w"'] & (-) \end{tikzcd} . \tag{$\star$}$$

Treating $w\mathscr{C}$ as a bicategory with a single vertical morphism reveals that

$$w\mathscr{C} \simeq v_{(-)}w_{(-)}\mathscr{C}.$$

This yields $wS_n\mathscr{C} \simeq v_{(-)}w_{(-)}S_n\mathscr{C}$ for each n.

Now, let $v_{(-)}\operatorname{co}w_{(-)}\mathscr{C}$ denote the subcategory of all squares like ($\star$) where the horizontal maps are also cofibrations. One can show that the inclusion $v_{(-)}\operatorname{co}w_{(-)}\mathscr{C} \subset v_{(-)}w_{(-)}\mathscr{C}$ is a homotopy equivalence. Since each $S_n\mathscr{C}$ inherits a cylinder functor from $\mathscr{C}$, we obtain a simplicial bi-subcategory $v_{(-)}\operatorname{co}w_{(-)}S_\bullet\mathscr{C}$ such that the inclusion into $v_{(-)}w_{(-)}S_\bullet\mathscr{C}$ is a homotopy equivalence. This yields a commutative diagram

$$\begin{tikzcd} vS_\bullet C^w \arrow[r] & vS_\bullet C \arrow[r] \arrow[d] & v_{(-)}\operatorname{co}w_{(-)}S_\bullet C \arrow[d, "\simeq"] \\ & wS_\bullet C \arrow[r, "\simeq"'] & v_{(-)}w_{(-)}S_\bullet C \end{tikzcd} .$$

What about the left vertical morphism?

It therefore suffices to show that the top row is a fibration. One can do this by using the relative K-theory space construction. See IV.8.5.3 and V.2.1 (Weibel). $\square$

Now, let $\mathscr{A}$ be an exact category embedded in an abelian category $\mathscr{B}$ and let $\mathbf{Ch}^{\mathrm{b}}(\mathscr{A})$ denote the category of bounded chain complexes in $\mathscr{A}$. One can verify that $\mathbf{Ch}^{\mathrm{b}}(\mathscr{A})$ is Waldhausen where the cofibrations $A_\bullet \rightarrowtail B_\bullet$ are precisely the degree-wise admissible monomorphisms (i.e., those admitting a short exact sequence $A_n \to B_n \to {}^{B_n}\!/_{A_n}$ in $\mathscr{A}$ for each n) and the weak equivalences are precisely the chain maps which are quasi-isomorphisms of complexes in $\mathbf{Ch}(\mathscr{B})$.

Our next result is a consequence of Theorem 2.1 and can be found in V.2.2 (Weibel).

Theorem 2.2 (Gillet-Waldhausen). *Let $\mathscr{A}$ be an exact category closed under kernels of surjections. Then the exact inclusion $\mathscr{A} \to \mathbf{Ch}^{\mathrm{b}}(\mathscr{A})$ induces a homotopy equivalence $K(\mathscr{A}) \simeq K\,\mathbf{Ch}^{\mathrm{b}}(\mathscr{A})$. Hence*

$$K_i(\mathscr{A}) = K_i\,\mathbf{Ch}^{\mathrm{b}}(\mathscr{A})$$

for every i.

Definition 2.3. Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor between *Waldhausen* categories. We say that F satisfies the *approximate lifting property* if for any map $b : F(A) \to B$ in $\mathscr{B}$, there exists a map $a : A \to A'$ in $\mathscr{A}$ along with a weak equivalence $b' : F(A') \simeq B$ in $\mathscr{B}$ such that

$$\begin{tikzcd} F(A') \arrow[r, dashed, "\simeq"] & B \\ F(A) \arrow[u, dashed, "F(a)"] \arrow[ur, "b"'] & \end{tikzcd} .$$

commutes.

This means that F has the approximate lifting property just in case we can always lift it to a weak equivalence.

Proposition 2.4. *Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor between Waldhausen categories with the following properties.*

1. *F satisfies the approximate lifting property.*
2. *$\mathscr{A}$ admits a cylinder functor.*
3. *A morphism f in $\mathscr{A}$ is a weak equivalence iff $F(f)$ is a weak equivalence in $\mathscr{B}$.*

Then $wF : w\mathscr{A} \to w\mathscr{B}$ is a homotopy equivalence.

Corollary 2.5 (Waldhausen approximation). *With the same hypotheses as in Proposition 2.4, we have*

$$K(\mathscr{A}) \simeq K(\mathscr{B}).$$

Proof sketch. One can show that each functor $S_n\mathscr{A} \to S_n\mathscr{B}$ is exact and also has the approximate lifting property. Proposition 2.4 thus implies that the bisimplicial map $wS_\bullet\mathscr{A} \to wS_\bullet\mathscr{B}$ is a degree-wise homotopy equivalence, which is enough. □

Definition 2.6. Let $\mathscr{A}$ be an abelian category $\mathbf{Ch}(\mathscr{A})$ denote the category of chain complexes over $\mathscr{A}$. We say that a complex $C_\bullet$ is *homologically bounded* if only finitely many $H_i(C_j)$ are nonzero.

Notation. Let $\mathbf{Ch}^{\mathrm{hb}}_{\pm}$ denote the subcategory of bounded below (respectively, bounded above) complexes.

Example 2.7. Let $\mathscr{A}$ be an abelian category. One can show that the inclusions $\mathbf{Ch}^{\mathrm{b}}(\mathscr{A}) \subset \mathbf{Ch}^{\mathrm{hb}}_{-}(\mathscr{A})$ and $\mathbf{Ch}^{\mathrm{hb}}_{+}(\mathscr{A}) \subset \mathbf{Ch}^{\mathrm{hb}}(\mathscr{A})$ have the approximate lifting property. Also, the inclusions $\mathbf{Ch}^{\mathrm{b}}(\mathscr{A}) \subset \mathbf{Ch}^{\mathrm{hb}}_{+}(\mathscr{A})$ and $\mathbf{Ch}^{\mathrm{hb}}_{+}(\mathscr{A}) \subset \mathbf{Ch}^{\mathrm{hb}}(\mathscr{A})$ satisfy the dual of the approximate lifting property. Thus, we can apply Corollary 2.5 along with Theorem 2.2 to find that

$$K(\mathscr{A}) \simeq K\,\mathbf{Ch}^{\mathrm{b}}(\mathscr{A}) \simeq K\,\mathbf{Ch}^{\mathrm{hb}}_{-} \simeq K\,\mathbf{Ch}^{\mathrm{hb}}_{+}(\mathscr{A}) \simeq K\,\mathbf{Ch}^{\mathrm{hb}}(\mathscr{A}).$$

3 K-theory spectrum

Definition 3.1. A *symmetric spectrum* $\mathbf{X}$ in topological spaces is a sequence of based Σ_n-spaces (X_n) endowed with structure maps $\sigma : X_n \wedge S^1 \to X_{n+1}$ such that $\sigma^k : X_n \wedge S^k \to X_{n+k}$ is $(\Sigma_n \times \Sigma_k)$-equivariant for any $n, k \geq 0$, where $S^k \equiv \underbrace{S^1 \wedge \cdots \wedge S^1}_{k \text{ times}}$.

A map $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ of symmetric spectra is a sequence $(f_n : X_n \to Y_n)$ of based Σ_n-equivariant maps such that for each $n \geq 0$, the square

$$\begin{array}{ccc} X_n \wedge S^1 & \xrightarrow{f_n \wedge \mathrm{Id}} & Y_n \wedge S^1 \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ X_{n+1} & \xrightarrow[f_{n+1}]{} & Y_{n+1} \end{array}$$

commutes. Let Sp^Σ denote the category of symmetric spectra in topological spaces.

Definition 3.2. Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. The *external n-fold $S_\bullet$-construction* on $\mathscr{C}$ is the n-multisimplicial Waldhausen category

$$(S_\bullet \cdots S_\bullet \mathscr{C}, wS_\bullet \cdots S_\bullet \mathscr{C}) .$$

In multidegree $(q_1, \ldots, q_n)$, it has as objects the diagrams $X : \mathrm{Ar}[q_1] \times \cdots \times \mathrm{Ar}[q_n] \to \mathscr{C}$ such that

(i) $X((i_1, j_1), \ldots, (i_n, j_n)) = *$ when $i_k = j_k$ for some $1 \leq k \leq n$ and

(ii) $X(\ldots, (i_t, j_t), \ldots) \rightarrowtail X(\ldots, (i_t, k_t), \ldots) \twoheadrightarrow X(\ldots, (j_t, k_t), \ldots)$ is a cofiber sequence in the $(n-1)$-fold iterated $S_\bullet$-construction for any $i_t \leq j_t \leq k_t$ in $[q_t]$.

Definition 3.3. Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. The *internal n-fold $S_\bullet$-construction* on $\mathscr{C}$ is the simplicial Waldhausen category

$$\left(S_\bullet^{(n)} \mathscr{C}, wS_\bullet^{(n)} \mathscr{C} \right) .$$

It has as q-simplices the functor categories $(S_q \cdots S_q \mathscr{C}, wS_q \cdots S_q \mathscr{C})$ whose objects are precisely the $(\mathrm{Ar}[q])^n$-shaped diagrams $X : (\mathrm{Ar}[q])^n \to \mathscr{C}$ such that

(i) $X((i_1, j_1), \ldots, (i_n, j_n)) = *$ when $i_k = j_k$ for some $1 \leq k \leq n$.

(ii) $X(\ldots, (i_t, j_t), \ldots) \rightarrowtail X(\ldots, (i_t, k_t), \ldots) \twoheadrightarrow X(\ldots, (j_t, k_t), \ldots)$ is a cofiber sequence in the $(n-1)$-fold iterated $S_\bullet$-construction for any $i_t \leq j_t \leq k_t$ in $[q]$.

Note that Σ_n acts on $S_\bullet^{(n)} \mathscr{C}$ by the relation $(\pi \cdot X)(\ldots, (i_t, j_t), \ldots) = X(\ldots, (i_{\pi^{-1}(t)}, j_{\pi^{-1}(t)}), \ldots)$.

The *(symmetric) algebraic K-theory spectrum* $\mathbf{K}(\mathscr{C}, w)$ of a small Waldhausen category $(\mathscr{C}, w\mathscr{C})$ has n-th space

$$K(\mathscr{C}, w)_n \equiv \left| wS_\bullet^{(n)} \mathscr{C} \right|$$

based at $*$. There is a Σ_n-action on $K(\mathscr{C}, w)_n$ induced by permuting the order of the internal $S_\bullet$-constructions. Moreover, we have that

$$\left|wS_\bullet^{(n)}\mathscr{C}\right| \wedge S^1 \cong \left|wS_\bullet^{(n)}S_\bullet\mathscr{C}\right|^{(1)} \subset \left|wS_\bullet^{(n)}S_\bullet\mathscr{C}\right| \cong \left|wS_\bullet^{(n+1)}\mathscr{C}\right|$$

where $-^{(1)}$ denotes the 1-skeleton with respect to the last simplicial direction. This determines the structure map σ.

Note 3.4. σ^k is $(\Sigma_n \times \Sigma_k)$-invariant.

Theorem 3.5. *For any $i \geq 0$, we have that $K_i(\mathscr{C}, w) = \pi_{i+1}K(\mathscr{C}, w)_1 \cong \pi_i\mathbf{K}(\mathscr{C}, w)$.*[1]

This enables us to encode our algebraic K-theory in an infinite loop space.

[1] Lemma 8.7.4 (Rognes).