

Abstract

These notes are based on Davi Maximo’s lectures for the course “Geometric Analysis and Topology I” at UPenn along with John Lee’s *Introduction to Smooth Manifolds*, 2nd Ed. and Michael Spivak’s *A Comprehensive Introduction to Differential Geometry, Vol. 1*. Any mistake in what follows is my own.

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1 Smooth manifolds

1.1 Lecture 1

We want to make precise our notion of a (topological) space that locally looks like \mathbb{R}^n .

Definition 1.1.1. A space M is a (topological) n -dimensional manifold (or n -manifold) if it is

- (i) Hausdorff,
- (ii) second-countable, and
- (iii) locally Euclidean of dimension n , i.e., for any $x \in M$, there exist an open set $U \ni x$ and a homeomorphism $\varphi : U \rightarrow V$ for some open subset $V \subset \mathbb{R}^n$.

Condition (iii) holds if and only if U is homeomorphic to an open ball in \mathbb{R}^n or to \mathbb{R}^n itself.

Terminology. A 2-dimensional manifold is called a *surface*.

Definition 1.1.2. Let M be an n -manifold.

1. A *coordinate chart on M* is a pair (U, φ) where $U \subset M$ is open and φ is a homeomorphism

$$U \xrightarrow[\text{open}]{\cong} W \subset \mathbb{R}^n.$$

If W is an open ball, then we call U a *coordinate ball*.

2. If (U, φ) is a coordinate chart and $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ denotes the i -th projection map, then we call elements of the set $\{(\pi_1(\varphi(p)), \dots, \pi_n(\varphi(p))) \mid p \in U\}$ *local coordinates on U* .

Notation. We shall use the symbols x^i and x_i interchangeably for local coordinates.

Definition 1.1.3.

1. Given charts (U, φ) , (V, ψ) with $U \cap V \neq \emptyset$, we say that the two are C^k -compatible if the *transition map* $\psi \circ \varphi^{-1}$

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & \varphi(U \cap V) \\ & \searrow \psi & \downarrow \psi \circ \varphi^{-1} \\ & & \psi(U \cap V) \end{array}$$

is C^k .

2. A collection of charts $(U_\alpha, \varphi_\alpha)$ which covers a smooth manifold M and is pairwise C^k -compatible is called a C^k -atlas for M .

Example 1.1.4. Consider the global charts $(\mathbb{R}, x \mapsto x)$ and $(\mathbb{R}, x \mapsto x^3)$. Since $x \mapsto x^{\frac{1}{3}}$ is not differentiable at 0, these charts fail to form a C^1 -atlas on \mathbb{R} .

Definition 1.1.5. An atlas A is *maximal* if it contains every chart that is C^∞ - (or smoothly) compatible with every chart in A .

Proposition 1.1.6.

1. Every smooth atlas A is contained in a unique maximal atlas, namely the family of all charts that are smoothly compatible with every chart in A .
2. Two smooth atlases are contained in the same maximal atlas if and only if their union is also a smooth atlas.

This shows that an atlas is maximal if and only if it's maximal in the usual in the set-theoretic sense.

Definition 1.1.7. A manifold M is *smooth* if it admits a maximal smooth atlas, also known as a *smooth structure*.

By Proposition 1.1.6, it's enough to construct any smooth atlas for M to show that it's a smooth manifold.

An open problem is whether there is more than one smooth structure on \mathbb{S}^4 . This is known for each $n \neq 4$. For example, Milnor (1958) gave an affirmative answer for \mathbb{S}^7 .

1.2 Lecture 2

Proposition 1.2.1. *If M admits a smooth structure, then M admits uncountably many smooth structures.*

Remark 1.2.2.

1. There exists a 10-dimensional topological manifold that admits no smooth structure (Kervaire 1961).
2. Any 2- or 3-dimensional manifold admits a smooth structure.

Let us now look at several examples of smooth structures on topological manifolds.

Example 1.2.3.

- (1) Any (real) vector space V where of dimension $n < \infty$ has a canonical smooth structure as follows. Endow V with any norm, since all norms on a finite-dimensional space are equivalent and hence generate the same topology. Pick any basis $B := (b_1, \dots, b_n)$ of V . Define the isomorphism $T : V \rightarrow \mathbb{R}^n$ by $b_i \mapsto e_i$ where e_i denotes the i -th standard basis vector. This is also a diffeomorphism, implying that V is a topological manifold and that (V, T) is an atlas on V . If B' is any other basis of V and T' the corresponding isomorphism, then the transition map $T' \circ T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear isomorphism, hence a diffeomorphism. By Proposition 1.1.6(2), it follows that any two bases determine the same smooth structure on V .
- (2) The restriction of a smooth structure on a smooth manifold M to an open subset $U \subset M$ yields a smooth structure on U , which is called an *open submanifold*.

Note that the general linear group $\text{GL}(n, \mathbb{F})$ is an open subset of $M(n, \mathbb{F})$, which is an n^2 -manifold by Example 1.2.3(1). Indeed, $\text{GL}(n, \mathbb{F}) = \det^{-1}(\mathbb{F}^{-1})$, the preimage of an open set in \mathbb{F} . By Example 1.2.3(2), $\text{GL}(n, \mathbb{F})$ is an open submanifold.

Example 1.2.4.

- (1) Let $U \subset \mathbb{R}^n$ be open and $F : U \rightarrow \mathbb{R}^m$ be continuous. Let $\Gamma(F)$ denote the graph of F and $\pi_1 \upharpoonright_{\Gamma(F)}$ be the restriction of the projection map $(x, y) \mapsto x$. This is a homeomorphism $\Gamma(F) \xrightarrow{\cong} U$ with inverse given by $x \mapsto (x, f(x))$. Hence $(\Gamma(F), \pi_1 \upharpoonright_{\Gamma(F)})$ is a smooth atlas on $\Gamma(F)$.

- (2) For each $i \in \{1, 2, \dots, n+1\}$, let $U_i^+ := \{\vec{x} \in \mathbb{R}^{n+1} : x_i > 0\}$. Define U_i^- similarly, so that the U_i^\pm cover the n -sphere

$$\mathbb{S}^n := \{\vec{x} \in \mathbb{R}^{n+1} : |\vec{x}| = 1\}.$$

Define the map $f : B_1(0) \subset \mathbb{R}^n \rightarrow \mathbb{R}$ by $f(\vec{u}) = \sqrt{1 - |\vec{u}|^2}$. Define $x_i : B_1(0) \rightarrow \mathbb{R}$ by $f(x_1, \dots, \hat{x}_i, \dots, x_n)$. Then $\Gamma(x_i) = U_i^+ \cap \mathbb{S}^n$, and $\Gamma(-x_i) = U_i^- \cap \mathbb{S}^n$. Thanks to (1), these graphs with their corresponding projections form a smooth structure on \mathbb{S}^n .

- (3) Let $f : U \subset \mathbb{R}^m \rightarrow \mathbb{R}$ be smooth. For each $c \in \mathbb{R}$, let $M_c := f^{-1}(c)$. Assume that the total derivative $\nabla f(a)$ is nonzero for each $a \in M_c$. Then $f_{x_i}(a) \neq 0$ for some $1 \leq i \leq m$. By the implicit function theorem, there is some smooth function $F : \mathbb{R}^{m-1} \rightarrow \mathbb{R}$ given by $x_i = F(x_1, \dots, \hat{x}_i, \dots, x_m)$ on some neighborhood $U_a \subset \mathbb{R}^m$ of a such that $f^{-1}(c) \cap U_a$ equals the graph of F . This means that the open sets $f^{-1}(c) \cap U_a$ together with their graph coordinates define a smooth atlas on M_c .

Example 1.2.5 (Real projective space). For each $i \in \{1, 2, \dots, n+1\}$, let $\tilde{U}_i := \{\vec{x} \in \mathbb{R}^{n+1} : x_i \neq 0\}$. Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}\mathbb{P}^n$ be the quotient map and $U_i := \pi(\tilde{U}_i)$. Since \tilde{U}_i is saturated and open, we know that $\pi|_{\tilde{U}_i}$ is a quotient map.¹ Define $f_i : U_i \rightarrow \mathbb{R}^n$ by

$$[x_1, \dots, x_{n+1}] \mapsto \left(\frac{x_1}{x_i}, \dots, \frac{x^{i-1}}{x_i}, \frac{x^{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i} \right),$$

whose inverse is given by $(x_1, \dots, x_n) \mapsto [x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n]$. Since $f_i \circ \pi$ is continuous, so is f_i .² Hence f_i is a homeomorphism. It's easy to check that each transition $f_i \circ f_j^{-1}$ is smooth. Thus, (U_i, f_i) defines a smooth atlas on $\mathbb{R}\mathbb{P}^n$.

Exercise 1.2.6. Show that $\mathbb{R}\mathbb{P}^n$ is second countable and Hausdorff.

Proof. Recall that $\mathbb{S}^n / \sim \cong \mathbb{R}\mathbb{P}^n$ where $x \sim y$ if $y = -x$. Thus it suffices to show these properties are true of $P^n := \mathbb{S}^n / \sim$.

To this end, let $\mathcal{B} := \{V_n\}$ denote the usual countable basis of \mathbb{S}^n inherited from \mathbb{R}^{n+1} . If $p \in U \subset P^n$ is open, then $\pi^{-1}(U)$ is a neighborhood of $\pi^{-1}(p)$, which equals $\{a, -a\}$ for some point a on the sphere. There exist $q \in \mathbb{Q}$ and $r \in \mathbb{Q}^{n+1}$ such that $\mathcal{B} \ni B_q(r) \cap \mathbb{S}^n \ni a$. In this case, $\mathcal{B} \ni B_q(-r) \cap \mathbb{S}^n \ni -a$. Note that the union of these two balls is contained in $\pi^{-1}(U)$ and is saturated, hence is mapped to a neighborhood $N \subset U$ of p . Thus $\{\pi(V_n)\}_{n \in \mathbb{N}}$ is a countable basis of P^n .

Proving that $\mathbb{R}\mathbb{P}^n$ is Hausdorff is quite similar. □

Example 1.2.7 (Product manifold). Let $M_1 \times \dots \times M_k$ be a product of n_i -dimensional smooth manifolds. Then this is a smooth manifold of dimension $n_1 + \dots + n_k$.

Lemma 1.2.8 (Smooth manifold construction). Let M be a set and let $\{U_\alpha\}$ be a collection of subsets equipped with injections $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ such that

(i) countably many U_α cover M ,

(ii) each $\varphi_\alpha(U_\alpha)$ is open,

¹Munkres, James. *Topology*. Theorem 22.1.

²Ibid. Theorem 22.2.

- (iii) any set of the form $\varphi_\alpha(U_\alpha \cap U_\beta)$ or $\varphi_\beta(U_\alpha \cap U_\beta)$ is open,
- (iv) if $U_\alpha \cap U_\beta \neq \emptyset$, then $\varphi_\alpha \circ \varphi_\beta^{-1}$ is smooth, and
- (v) if $p, q \in M$ with $p \neq q$, then either both are in U_α for some α or they can be separated by sets in $\{U_\alpha\}$.

Then M has a unique smooth manifold structure with $(U_\alpha, \varphi_\alpha)$ as charts.

Notation. The expression M^n means that M is an n -dimensional manifold.

Definition 1.2.9. If $f : M^n \rightarrow \mathbb{R}$ is a function with M smooth, we say that f is *differentiable at p* if there is some chart $(U_\alpha, \varphi_\alpha)$ such that the coordinate representation $f \circ \varphi_\alpha^{-1} : \varphi(U_\alpha) \rightarrow \mathbb{R}$ is differentiable at p .

We must ensure that Definition 1.2.9 is coordinate-independent.

Lemma 1.2.10. If $f \circ \varphi^{-1}$ is differentiable at $\varphi(p)$ and $\psi : V \rightarrow \mathbb{R}^n$ is another coordinate neighborhood of $p \in M^n$, then $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}$ is also differentiable at $\varphi(p)$.

Proof. This holds because

$$\begin{array}{ccc}
 U \cap V & \xrightarrow{f} & \mathbb{R} \\
 \varphi \downarrow & \nearrow f \circ \varphi^{-1} & \uparrow f \circ \psi^{-1} \\
 \varphi(U \cap V) & \xleftarrow{\varphi \circ \psi^{-1}} & \psi(U \cap V)
 \end{array}$$

commutes. □

2 Smooth maps

2.1 Lecture 3

Definition 2.1.1. Let M^n and N^k be smooth manifolds. We say that $F : M \rightarrow N$ is *smooth at $p \in M$* if there are charts $(V, \varphi) \ni p$ and $(V', \psi) \ni F(p)$ with $F(V) \subset V'$ such that the coordinate representation $\psi \circ F \circ \varphi^{-1}$ is smooth.

$$\begin{array}{ccc}
 V & \xrightarrow{F} & V' \\
 \varphi \downarrow & & \downarrow \psi \\
 \varphi(V) & \xrightarrow{\psi \circ F \circ \varphi^{-1}} & \psi(V')
 \end{array}$$

This definition is independent of coordinates. Indeed, if $(U, \bar{\varphi})$ and $(U', \bar{\psi})$ are other charts around p and $F(p)$, respectively, then

$$\begin{aligned}
 \bar{\psi} \circ F \circ \varphi^{-1} &= (\bar{\psi} \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}) \\
 \bar{\psi} \circ F \circ \bar{\varphi}^{-1} &= (\bar{\psi} \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ \bar{\varphi}^{-1}),
 \end{aligned}$$

which are smooth as composites of smooth maps.

Lemma 2.1.2. *Smoothness implies continuity.*

Proof. Using notation as in Definition 2.1.1, we see that for each $p \in M$, there is a neighborhood V of p such that $F|_V = \psi^{-1} \circ \psi \circ F \circ \varphi^{-1} \circ \varphi$ is a composite of continuous maps (as we know smoothness implies continuity for maps between Euclidean spaces) and thus itself continuous. We can glue these restrictions together to conclude that F is continuous. \square

Note 2.1.3. Being smooth is a local property of maps.

1. Given $F : M \rightarrow N$, if every $p \in M$ has a neighborhood U_p so that $F|_{U_p}$ is smooth, then F is smooth.
2. Conversely, the restriction of any smooth map to an open subset is smooth.

Example 2.1.4. The natural projection $\pi : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}\mathbb{P}^n$ is smooth. Let $v \in (\mathbb{R}^{n+1} \setminus \{0\}, \text{id})$. Let $(U_i, \varphi_i) \in A_n$ be a neighborhood of $\pi(p)$. Since π is continuous, $S := \pi^{-1}(U_i) \cap (\mathbb{R}^{n+1} \setminus \{0\})$ is a neighborhood of v . Further, $\varphi_i \circ \pi \circ \text{id} : S \rightarrow \varphi_i(U_i)$ is given by $x \mapsto \frac{(x_1, \dots, \hat{x}_i, \dots, x_{n+1})}{x_i}$, which is smooth.

Definition 2.1.5. A smooth map with a smooth inverse is a *diffeomorphism*.

This defines an equivalence relation \approx between smooth manifolds. Thanks to Lemma 2.1.2, any diffeomorphism is a homeomorphism, which gives us the following result.

Theorem 2.1.6. *If $M^n \approx N^k$, then $n = k$.*

Example 2.1.7.

1. $(\mathbb{R}, \text{id}) \approx (\mathbb{R}, x \mapsto x^{\frac{1}{3}})$ via the mapping $x \mapsto x^3$.
2. $F : \mathbb{B}^n \rightarrow \mathbb{R}^n$ given by $F(x) = \frac{x}{\sqrt{1-|x|^2}}$ is a diffeomorphism with inverse $G(y) = \frac{y}{\sqrt{1+|y|^2}}$.
3. $\mathbb{S}^n / \sim \approx \mathbb{R}\mathbb{P}^n$.
4. If M is a smooth manifold and (U, φ) is a chart, then $\varphi : U \rightarrow \varphi(U)$ is a diffeomorphism.

At this point, we want to develop tools with which we can glue together already locally defined smooth functions $U_\alpha \rightarrow \mathbb{R}$ to obtain a globally defined smooth function $M \rightarrow \mathbb{R}$.

Definition 2.1.8. If M is any space and $f : M \rightarrow \mathbb{R}^n$ is continuous, then the *support of f* is

$$\text{supp } f := \text{cl}(\{x \in M : f(x) \neq 0\}).$$

Lemma 2.1.9. *Given any $0 < r_1 < r_2$, there is some smooth function $H : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

- $H = 1$ on $\bar{B}_{r_1}(0)$,
- $0 < H < 1$ on $B_{r_2}(0) \setminus \bar{B}_{r_1}(0)$, and
- $H = 1$ elsewhere.

Proof. We construct such an H . First recall that $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

is smooth. Now define $h : \mathbb{R} \rightarrow \mathbb{R}$ by $h(t) = \frac{f(r_2-t)}{f(r_2-t)+f(t-r_1)}$. Finally, define $H : \mathbb{R}^n \rightarrow \mathbb{R}$ by $H(x) = h(|x|)$. \square

2.2 Lecture 4

Definition 2.2.1. Let \mathcal{U} and \mathcal{V} be open covers of a space X .

1. \mathcal{V} is a *refinement* of \mathcal{U} if for every $V \in \mathcal{V}$, there is some $U \in \mathcal{U}$ such that $V \subset U$.
2. \mathcal{U} is *locally finite* if each $x \in X$ has some neighborhood that intersects only finitely many $U \in \mathcal{U}$.
3. X is *paracompact* if every open cover of X admits a locally finite refinement.

We are now ready to define our main tool for patching together local functions to obtain a global one.

Definition 2.2.2. Let M be a space and $\mathcal{X} := (X_\alpha)_{\alpha \in A}$ be an open cover. A *partition of unity subordinate to \mathcal{X}* is a family $(\psi_\alpha)_{\alpha \in A}$ of continuous functions $\psi_\alpha : M \rightarrow \mathbb{R}$ with the following properties.

- (a) $0 \leq \psi_\alpha(x) \leq 1$ for each α and x .
- (b) $\text{supp } \psi_\alpha \subset X_\alpha$ for each α .
- (c) The family $(\text{supp } \psi_\alpha)$ is locally finite, in the sense that every point $p \in M$ has a neighborhood V_p such that $V_p \cap \text{supp } \psi_\alpha \neq \emptyset$ for at most finitely many α . In particular, M is paracompact.
- (d) $\sum_{\alpha \in A} \psi_\alpha(x) \equiv \sup \left\{ \sum_{\alpha \in F} \psi_\alpha(x) : \substack{F \subset A \\ \text{finite}} \right\} = 1$ for each x .

Lemma 2.2.3. *Every topological manifold M is paracompact.*

Before proving this, let us recall that a subspace is *precompact* if its closure is compact.

Proof. Since M has a countable atlas, it has a countable basis $\{B_n\}$ of precompact coordinate balls. (The continuous image of a precompact set into a Hausdorff space is also precompact.)

Step 1: By induction, we can build a countable covering $\{U_n\}$ of precompact sets such that $\text{cl}(U_{n-1}) \subset U_n$ and $B_n \subset U_n$ for each n .

Step 2: We build a countable locally finite open cover $\{V_n\}$. Let

$$V_n = \begin{cases} \text{cl}(U_n) \setminus U_{n-2} & n > 2 \\ V_n = U_n & \text{otherwise} \end{cases}.$$

Note that every V_n intersects only finitely many other V_j , hence $\{V_n\}$ is locally finite.

Step 3: Let $\{X_\alpha\}$ be any open cover. For any $p \in M$, there is some α with $p \in X_\alpha$ as well as some neighborhood W_p that intersects V_j for only finitely many $j \in \mathbb{N}$. Set $\widetilde{W}_p = W_p \cap X_\alpha$. Then the \widetilde{W}_p cover M . Since each V_j is precompact by construction, we know that V_j has a finite subcover $\widetilde{W}_{p_{j_{k_1}}}, \dots, \widetilde{W}_{p_{j_{k_j}}}$. Then

$$V_j = \left(V_j \cap \widetilde{W}_{p_{j_{k_1}}} \right) \cup \dots \cup \left(V_j \cap \widetilde{W}_{p_{j_{k_j}}} \right),$$

and thus $\left\{ \left(V_j \cap \widetilde{W}_{p_{j_{k_1}}} \right), \dots, \left(V_j \cap \widetilde{W}_{p_{j_{k_j}}} \right) \right\}_{j \in \mathbb{N}}$ is a locally finite refinement of $\{X_\alpha\}$, as desired. \square

Remark 2.2.4. If X is connected, then X is paracompact if and only if it is second-countable.

Theorem 2.2.5 (Existence of partition of unity). *If M is a smooth manifold, then any open cover $\mathcal{X} := \{X_\alpha\}_{\alpha \in A}$ of M admits a partition of unity.*

Proof. For each $\alpha \in A$, we can find a countable basis \mathcal{C}_α of precompact coordinate balls centered at 0 for X_α . Then $\mathcal{C} := \bigcup_\alpha \mathcal{C}_\alpha$ is a basis for M . Since M is paracompact, \mathcal{X} admits a locally finite refinement $\{C_i\}_{i \in \mathbb{I}}$ consisting of elements of \mathcal{C} . Note that the cover $\{\text{cl}(B_i)\}$ is also locally finite. There are coordinate balls $C'_i \subset X_{\alpha_i}$ such that $C'_i \supset \text{cl}(C_i)$. For each $i \in \mathbb{I}$, let $\varphi_i : C'_i \rightarrow \mathbb{R}^n$ be a smooth coordinate map so that $\varphi_i(C'_i) \supset \varphi(C_i)$ and $\varphi(\text{cl}(C_i)) = \text{cl}(\varphi(C_i))$. Define $f_i : M \rightarrow \mathbb{R}$ by

$$f_i(x) = \begin{cases} H_i \circ \varphi_i & x \in C'_i \\ 0 & x \in M \setminus \text{cl}(C_i) \end{cases}$$

where $H_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is as in Lemma 2.1.9: a smooth function that is positive on $\varphi_i(C_i)$ and zero elsewhere. Note that f_i is well-defined because $f_i = 0$ on $C'_i \setminus \text{cl}(C_i)$. Also, it is smooth by the point-set gluing lemma for open sets.

Define $f : M \rightarrow \mathbb{R}$ by $f(x) = \sum_i f_i(x)$, which is a finite sum and hence well-defined. We see that f is a smooth function and that $f(x) > 0$ for each $x \in M$. Then $g_i(x) \equiv \frac{f_i(x)}{f(x)}$ defines a smooth function $M \rightarrow \mathbb{R}$ for each i , so that $\sum_i g_i(x) = 1$ and $0 \leq g_i(x) \leq 1$ for each $x \in M$. Note that $\text{supp}(g_i) = \text{cl}(C_i)$.

For each $\alpha \in A$, define $\psi_\alpha : M \rightarrow \mathbb{R}$ by

$$\psi_\alpha(x) = \sum_{\substack{i \\ \alpha_i = \alpha}} g_i(x).$$

Interpret this as the zero function when there are no i such that $\alpha_i = \alpha$. Note that each ψ_α is smooth as a finite sum of smooth functions and satisfies $0 \leq \psi_\alpha \leq 1$. Moreover, we have that

$$\text{supp}(\psi_\alpha) = \text{cl} \left(\bigcup_{\substack{i \\ \alpha_i = \alpha}} C_i \right) = \bigcup_{\substack{i \\ \alpha_i = \alpha}} \text{cl}(C_i).$$

Since $\{\text{cl}(C_i)\}$ is locally finite, so is $\{\text{supp}(\psi_\alpha)\}_{\alpha \in A}$. Finally, the fact that $\alpha_i \in A$ implies that

$$\sum_\alpha \psi_\alpha(x) = \sum_i g_i(x) = 1$$

for each $x \in M$. Therefore, we may take $\{\psi_\alpha\}$ as our desired partition of unity. \square

Corollary 2.2.6 (Bump function). *If $A \subset U \subset M$ with A closed and U open in M , then there is a smooth function $f : M \rightarrow \mathbb{R}$ such that $f(x) = 1$ for each $x \in A$ and $f(x) = 0$ outside a neighborhood of A .*

Proof. Since $\{U, M \setminus A\}$ is an open cover of M , there is a partition of unity φ_1, φ_2 such that $\text{supp} \varphi_1 \subset U$, $\text{supp} \varphi_2 \subset M \setminus A$, and $\varphi_1 + \varphi_2 = 1$. Hence $\varphi_1 \upharpoonright_A = 1 - 0 = 1$, and $\varphi_1 \upharpoonright_{M \setminus U} = 0$. \square

2.3 Lecture 5

Corollary 2.3.1 (Whitney). *Let M be a smooth manifold and $K \subset M$ be closed. Then there exists a non-negative smooth function $f : M \rightarrow \mathbb{R}$ such that $f^{-1}(0) = K$.*

This means that closed subsets of smooth manifolds are completely characterized as the 0-level sets of smooth maps. Being the 0-level set of analytic maps, such as polynomials, is much more special. Any object with such a property is called an *analytic submanifold* and is studied in algebraic geometry.

Proof. First assume that $M = \mathbb{R}^n$. We have that $M \setminus K$ is open, which is thus the union of countably many balls $B_{r_i}(x_i)$ with $r_i \leq 1$. Construct, as in Lemma 2.1.9, a smooth bump function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $h(x) = 1$ on $\bar{B}_{\frac{1}{2}}(0)$ and h is supported in $B_1(0)$. By our construction of h , we can verify that for each $i \in \mathbb{N}$, there is some $C_i \geq 1$ that bounds any of the partials of h up through order i .

Define $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\sum_{i=1}^{\infty} \frac{r_i^i}{2^i C_i} h\left(\frac{x - x_i}{r_i}\right).$$

Each i -th term is bounded by $\frac{1}{2^i}$. Thanks to the Weierstrass M-test, f is well-defined and continuous. Since h is zero outside $B_1(0)$, we see that $f^{-1}(0) = K$.

To see that f is smooth, assume by induction that f is C^{k-1} for a given $k \geq 1$. By the chain rule and induction, we can write any k -th partial D_k of the i -th term of the series defining f as $\frac{(r_i)^{i-k}}{2^i C_i} D_k h\left(\frac{x-x_i}{r_i}\right)$. As h is smooth, this expression is C^1 . And since $r_i \leq 1$ and C_i bounds all partials up to order i , it is eventually bounded by $\frac{1}{2^i}$. Hence the series of these expressions converges uniformly to a continuous function. By Theorem C.31 (Lee), it follows that $D_k f$ exists and is continuous, thereby completing our induction.

Now, assume that M is arbitrary. Find a cover (B_α) of smooth coordinate balls for M . Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to this cover. Note that each B_α is diffeomorphic to \mathbb{R}^n . Since the property of admitting a non-negative smooth function $f : M \rightarrow \mathbb{R}$ with $f^{-1}(0) = K$ can be stated in the language of smooth manifolds, it is invariant under diffeomorphism. Thus, there is some non-negative smooth function $f_\alpha : B_\alpha \rightarrow \mathbb{R}$ where $f_\alpha^{-1}(0) = K \cap B_\alpha$ for each α . Then it's straightforward to check that $g \equiv \sum_\alpha \varphi_\alpha f_\alpha$ is as desired. \square

Corollary 2.3.2. *Let M be a smooth manifold and $K \subset M$ be closed. Let $c > 0$. Then there exists a non-negative smooth function $f : M \rightarrow \mathbb{R}$ such that $f^{-1}(c) = K$.*

Exercise 2.3.3. *Prove that the restriction of a smooth map on \mathbb{R}^{n+1} to \mathbb{S}^n is smooth.*

3 Tangent vectors

3.1 Lecture 6

We can view the tangent space $T_p \mathbb{S}^n$ of \mathbb{S}^n at a point p as all of the directions from p with respect to which you can find the rate of change of a smooth map f provided that you're only allowed to roam through \mathbb{S}^n . We want to generalize our notion of a tangent space to arbitrary manifolds in order to do first-order calculus on them.

Notation. We shall denote the space of smooth functions $M \rightarrow \mathbb{R}$ by $C^\infty(M)$.

Definition 3.1.1. Given $a \in \mathbb{R}^n$, a map $\omega : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ is called a *derivation at a* if it

- (i) is linear over \mathbb{R} and

(ii) satisfies the *Leibniz rule*:

$$\omega(fg) = f(a)\omega(g) + g(a)\omega(f)$$

for any $f, g \in C^\infty(\mathbb{R}^n)$.

Let $T_a\mathbb{R}^n$ denote the vector space of derivations at a .

Note 3.1.2. If f is constant, then $\omega f = 0$ for any derivation ω .

Example 3.1.3. For any $u \in \mathbb{R}^n$, recall that the directional derivative of $f \in C^\infty(\mathbb{R}^n)$ in the direction u at a is

$$D_u f(a) \equiv \lim_{h \rightarrow 0} \frac{1}{h} (f(a + hu) - f(a)) = \left. \frac{d}{dh} \right|_{h=0} f(a + hu).$$

Then this is a derivation of f at a .

Notation. For any $a \in \mathbb{R}^n$, let \mathbb{R}_a^n denote the (real) vector space $\{(a, v) \mid v \in \mathbb{R}^n\}$.

Theorem 3.1.4. For each $a \in \mathbb{R}^n$, define $L_a : \mathbb{R}_a^n \rightarrow T_a\mathbb{R}^n$ by $v_a \mapsto D_v|_a$. This is an isomorphism.

Proof. It is clear that L_a is linear. It remains to show that it is both injective and surjective.

Suppose that $u, v \in \mathbb{R}_a^n$ and $L_a(u) = L_a(v)$. Then by linearity $L_a(u - v) = 0$, yielding

$$\left. \frac{d}{dt} \right|_{t=0} f(a + t(u - v)) = 0$$

for any smooth function f . But if $u - v \neq 0$, then this says that for any f , the directional derivative of f at a in the direction of a certain nonzero vector vanishes, which is clearly false. Hence $u = v$, and L_a is injective.

Next, suppose that $\omega \in T_a\mathbb{R}^n$ and consider the coordinate projection $x^i : \mathbb{R}^n \rightarrow \mathbb{R}$ for each $i = 1, \dots, n$. Set $v_i = \omega(x^i)$ and write $v = v_i e_i$. We claim that $L_a(v) = D_v|_a = \omega$. By Taylor's theorem, any $f \in C^\infty(\mathbb{R}^n)$ has an expansion

$$f(x) = f(a) + \sum_{i=1}^n f_{x_i}(a)(x_i - a_i) + c \sum_{i,j=1}^n (x_i - a_i)(x_j - a_j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(a + t(x - a)) dt$$

for some $c > 0$. Each term of the second sum is the product of two smooth functions vanishing at a . We can apply the product rule along with linearity of ω to conclude that

$$\begin{aligned} \omega f &= \omega \left(\sum_{i=1}^n f_{x_i}(a)(x_i - a_i) \right) \\ &= \sum_{i=1}^n \omega(f_{x_i}(a)(x_i - a_i)) \\ &= \sum_{i=1}^n f_{x_i}(a)(\omega(x_i) - \omega(a_i)) \\ &= \sum_{i=1}^n f_{x_i}(a)v_i \\ &= D_v|_a f. \end{aligned}$$

□

Corollary 3.1.5. We have $\dim(T_a\mathbb{R}^n) = n$, and the partial derivatives $\left\{ \frac{\partial}{\partial x_i} \Big|_a \right\}_{1 \leq i \leq n}$ form a basis of $T_a\mathbb{R}^n$.

Definition 3.1.6. Let M be a smooth manifold and let $p \in M$.

1. An \mathbb{R} -linear map $v : C^\infty(M) \rightarrow \mathbb{R}$ is called a *derivation at p* if

$$v(fg) = f(p)v(g) + v(f)g(p)$$

for any f and g .

2. The tangent space of M at p is the vector space

$$T_pM \equiv \{ \omega : C^\infty(M) \rightarrow \mathbb{R} : \omega \text{ is a derivation of } M \text{ at } p \}.$$

Any element of this space is called a *tangent vector*.

Definition 3.1.7 (Differential of a smooth map). Given smooth manifolds M and N , a smooth map $F : M \rightarrow N$, and $p \in M$, we define the *differential of F at p* as the map $dF_p : T_pM \rightarrow T_{F(p)}N$ given by

$$dF_p(v)(f) = v(f \circ F).$$

Terminology. We call $dF_p(v)$ the *pushforward of v by dF* .

Proposition 3.1.8. Let M , N , and P be smooth manifolds, $F : M \rightarrow N$ and $G : N \rightarrow P$ be smooth maps, and $p \in M$.

1. $dF_p : T_pM \rightarrow T_{F(p)}N$ is linear.
2. $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \rightarrow T_{G(F(p))}P$.
3. $d(\text{id}_M)_p = \text{id} : T_pM \rightarrow T_pM$.
4. If F is a diffeomorphism, then dF_p is an isomorphism with inverse $d(F^{-1})_{F(p)}$.

Aside. This shows that mapping (M, p) to T_pM and $F : (M, p) \rightarrow (N, F(p))$ to dF_p defines a functor from \mathbf{Diff}_* to $\mathbf{Vec}_{\mathbb{R}}$, known as the *tangent space functor*.

Lemma 3.1.9. Let $v \in T_pM$ and $f, g \in C^\infty(M)$. Then if f and g agree on a neighborhood N_p of p , then $vg = vf$.

Proof. Set $h = f - g$, so that h vanishes on N_p . We can find a smooth bump function $\varphi : M \rightarrow \mathbb{R}$ such that $\varphi \equiv 1$ on $\text{supp}(h)$ and $\text{supp}(\varphi) \subset M \setminus \{p\}$. Then $\varphi h(x) = h(x)$ for any $x \in M$. Since both φ and h vanish at p , it follows that $vf - vg = v\varphi h = v(\varphi h) = 0$. \square

Proposition 3.1.10. If M is an n -dimensional smooth manifold, then $\dim(T_pM) = n$ for every $p \in M$.

In particular, we identify the standard basis $\{e_1, \dots, e_n\}$ for \mathbb{R}^n by $e_i \leftrightarrow \left(0, \dots, 0, \frac{\partial}{\partial x_i} \Big|_p, 0, \dots, 0 \right)$.

3.2 Lecture 7

Given a point $p \in M$, find a chart $(U, \varphi) \ni p$. Then $d\varphi_p : T_p M \cong T_p U \rightarrow T_{\varphi(p)} \varphi(U) \cong T_p \mathbb{R}^n$ is an isomorphism. This choice of chart yields a natural choice of basis for $T_p M$:

$$\left\{ \frac{\partial}{\partial x_i} \Big|_p \right\}_{1 \leq i \leq n}$$

where

$$\frac{\partial}{\partial x_i} \Big|_p := (d\varphi_p)^{-1} \left(\frac{\partial}{\partial x_i} \Big|_{\varphi(p)} \right) = (d\varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x_i} \Big|_{\varphi(p)} \right). \quad (*)$$

Let $F : M \rightarrow N$ be smooth with $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^m$ open. Then by the chain rule we get

$$\begin{aligned} dF_p \left(\frac{\partial}{\partial x_i} \Big|_p \right) f &= \frac{\partial}{\partial x_i} \Big|_p (f \circ F) \\ &= \frac{\partial}{\partial x_i} \Big|_p (f(F_1, \dots, F_m)) \\ &= \sum_{j=1}^m \frac{\partial f}{\partial F_j} (F(p)) \frac{\partial F_j}{\partial x_i} (p) \\ &= \sum_{j=1}^m \frac{\partial F_j}{\partial x_i} (p) \left(\frac{\partial}{\partial y_j} \Big|_{F(p)} \right) f. \end{aligned}$$

Therefore, dF_p can be represented by the familiar $m \times n$ Jacobian matrix of F at p ,

$$DF(p) := \begin{bmatrix} \vdots & & \vdots \\ \frac{\partial F_j}{\partial x_1} (p) & \cdots & \frac{\partial F_j}{\partial x_n} (p) \\ \vdots & & \vdots \end{bmatrix},$$

which acts on $\mathbb{R}^n \cong T_p M$.

Now consider the general case $F : M \rightarrow N$ smooth between manifolds. For any $p \in M$, choose charts $(U, \varphi) \ni p$ and $(V, \psi) \ni F(p)$. Then the Euclidean map $\widehat{F} := \psi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U) \rightarrow \psi(V)$ is smooth. If $\widehat{p} := \varphi(p)$, it follows from $(*)$ that $d\widehat{F}_{\widehat{p}}$ is represented by the Jacobian of \widehat{F} at \widehat{p} . Noting that $F \circ \varphi^{-1} = \psi^{-1} \circ \widehat{F}$, we compute

$$\begin{aligned} dF_p \left(\frac{\partial}{\partial x_i} \Big|_p \right) &= dF_p \left(d(\varphi^{-1}) \Big|_{\widehat{p}} \left(\frac{\partial}{\partial x_i} \Big|_{\widehat{p}} \right) \right) \\ &= d(\psi^{-1}) \Big|_{\widehat{F}(\widehat{p})} \left(d\widehat{F} \Big|_{\widehat{p}} \left(\frac{\partial}{\partial x_i} \Big|_{\widehat{p}} \right) \right) \\ &= d(\psi^{-1}) \Big|_{\widehat{F}(\widehat{p})} \left(\sum_{j=1}^m \frac{\partial \widehat{F}_j}{\partial x_i} (\widehat{p}) \frac{\partial}{\partial y_j} \Big|_{\widehat{F}(\widehat{p})} \right) \\ &= \sum_{j=1}^m \frac{\partial \widehat{F}_j}{\partial x_i} (\widehat{p}) \frac{\partial}{\partial y_j} \Big|_{F(p)}. \end{aligned}$$

Therefore, dF_p can be represented by the Jacobian matrix of \widehat{F} at \widehat{p} .

Given any two pairs of coordinates for p and $F(p)$, the respective Jacobian matrices are related by the familiar change-of-basis matrix. In particular, they are similar.

Given a smooth manifold M , we define a notion of a smoothly varying tangent space as follows.

Definition 3.2.1. The *tangent bundle* of M is the set

$$TM \equiv \coprod_{p \in M} T_p M$$

endowed with a certain natural topology induced by the projection $\pi : TM \rightarrow M$, $(\varphi, p) \mapsto p$.

Example 3.2.2. As \mathbb{R}_a^n is canonically isomorphic to \mathbb{R}^n , we have $T\mathbb{R}^n \cong \coprod_{a \in \mathbb{R}^n} \{a\} \times \mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$.

3.3 Lecture 8

Lemma 3.3.1. For any smooth n -dimensional manifold M , the tangent bundle TM has a natural topology and smooth structure such that

- TM is a $2n$ -dimensional smooth manifold and
- the projection $\pi : TM \rightarrow M$ is smooth.

Proof. Given a chart (U, φ) , define $\tilde{\varphi} : \pi^{-1}(U) \rightarrow \mathbb{R}^n$ by

$$v_i \frac{\partial}{\partial x_i} \Big|_p \mapsto (x^1(p), \dots, x^n(p), v_1, \dots, v_n)$$

where $\varphi = (x^1, \dots, x^n)$.³ This is continuous with $\text{Im } \tilde{\varphi} = \varphi(U) \times \mathbb{R}^n$, which is open. Further, $\tilde{\varphi}^{-1}$ is given by $(x_1, \dots, x_n, v_1, \dots, v_n) \mapsto v_i \frac{\partial}{\partial x_i} \Big|_{\varphi^{-1}(x)}$ on $\varphi(U) \times \mathbb{R}^n$. Take $\{(\pi^{-1}(U), \tilde{\varphi})\}$ to be charts on TM . Given two such charts $(\pi^{-1}(U), \tilde{\varphi})$ and $(\pi^{-1}(V), \tilde{\psi})$, it's straightforward to check that $\tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \rightarrow \psi(U \cap V) \times \mathbb{R}^n$ is smooth.

Next, notice that if we take a countable cover $\{U_i\}$ of M by smooth coordinate domains, then $\{\pi^{-1}(U_i)\}$ satisfies the conditions of Lemma 1.2.8.

Finally, to see that $\pi : TM \rightarrow M$ is smooth, notice that its coordinate representation at every point is given by the projection $\pi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$, $(x, v) \mapsto x$. \square

Terminology. We call the $\tilde{\varphi}((f, p))$ the *natural coordinates* on TM .

Given $F : M \rightarrow N$ is smooth, define the *global differential* $dF : TM \rightarrow TN$ of F by $dF(\varphi, p) = dF_p(\varphi)$.

Proposition 3.3.2. The global differential $dF : TM \rightarrow TN$ is smooth.

Aside. This shows that mapping M to TM and F to dF defines a functor from **Diff** to itself, known as the *tangent functor*.

Note 3.3.3. If F is a diffeomorphism, then so is dF with $d(F^{-1}) = (df)^{-1}$.

Definition 3.3.4. Given a smooth curve $\gamma : J \rightarrow M$ and $t_0 \in J$, the *velocity of γ at t_0* is

$$\gamma'(t_0) \equiv d\gamma \left(\frac{d}{dt} \Big|_{t_0} \right) \in T_{\gamma(t_0)} M.$$

³The expression $v_i \frac{\partial}{\partial x_i} \Big|_p$ is secretly a summation, in accordance with the *Einstein summation convention*.

Note 3.3.5. Let $(U, \varphi) \ni \gamma(t_0)$ be a chart on M . Then $\gamma'(t_0) = \frac{d\gamma^i}{dt} \frac{\partial}{\partial x_i} \Big|_{\gamma(t_0)}$.

Lemma 3.3.6. Every $v \in T_p M$ is the velocity of some smooth curve $\gamma : J \rightarrow M$ at 0 such that $\gamma(0) = p$.

Proof. Let (U, φ) be a chart centered at p . Write $v = v_i \frac{\partial}{\partial x_i} \Big|_p$. For any $\epsilon > 0$ small, define $\gamma : (-\epsilon, \epsilon) \rightarrow U$ by $\gamma(t) = \varphi^{-1}(tv_1, \dots, tv_n)$. Note 3.3.5 implies that $\gamma'(0) = v$. \square

Proposition 3.3.7. Let $v \in T_p M$. Then $dF_p(v) = (F \circ \gamma)'(0)$ for any smooth map $\gamma : J \rightarrow M$ satisfying $\gamma(0) = p$ and $\gamma'(0) = v$.

Aside. A smooth function element on M is a pair (f, U) with $U \subset M$ open and $f : M \rightarrow \mathbb{R}$ smooth. Say that $(f, U) \sim (g, V)$ if $p \in U \cap V$ and $f = g$ on some neighborhood of p . The equivalence class $[f]_p := [(f, U)]$ is called the *germ of f at p* . The set of such classes is denoted by $C_p^\infty(M)$. This is an associative algebra over \mathbb{R} .

Define a *derivation of $C_p^\infty(M)$* as a linear map $v : C_p^\infty(M) \rightarrow \mathbb{R}$ satisfying $v[fg]_p = f(p)v[g]_p + g(p)v[f]_p$. The tangent space $\mathcal{D}_p M$ of such derivations serves as an equivalent (in the sense of isomorphism) definition of the tangent space of M at p .

3.4 Lecture 9

Theorem 3.4.1 (Inverse function). If $F : M \rightarrow N$ is smooth and dF_p is invertible, then there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \rightarrow V_0$ is a diffeomorphism.

Proof. Notice that M and N have equal dimension (say n) because dF_p is invertible. Choose charts (U, f) centered at p and (V, g) centered at $F(p)$ such that $F(U) \subset V$. Then $\widehat{F} := g \circ F \circ f^{-1}$ is smooth map from $f(U) \subset \mathbb{R}^n$ to $g(V) \subset \mathbb{R}^n$ with $\widehat{F}(0) = 0$. Now $d\widehat{F}_0$ is invertible as the composite of three invertible maps. The inverse function theorem for Euclidean space implies that there are open balls $B_r(0)$ and $B_s(0)$ such that $\widehat{F} : B_r(0) \rightarrow B_s(0)$ is a diffeomorphism. Thus, we can take $F : f^{-1}(B_r(0)) \rightarrow g^{-1}(B_s(0))$ as our desired diffeomorphism. \square

Corollary 3.4.2. If dF_p is nonsingular at each $p \in M$, then F is a local diffeomorphism.

Proposition 3.4.3.

1. The finite product of local diffeomorphisms is a local diffeomorphism.
2. The composite of two local diffeomorphisms is a local diffeomorphism.
3. Any bijective local diffeomorphism is a diffeomorphism.
4. A map F is a local diffeomorphism if and only if each point in $\text{dom}(F)$ has a neighborhood where F 's coordinate representation is a local diffeomorphism.

Definition 3.4.4. The rank of a smooth map F at a point p is the rank of dF_p . If the rank of F is the same at each point, then we say F has constant rank.

Theorem 3.4.5 (Constant rank). Let $F : M^m \rightarrow N^n$ be smooth with constant rank $r \leq m, n$. Then for each $p \in M$, there are charts (U, f) centered at p and (V, g) centered at $F(p)$ such that $F(U) \subset V$ and the coordinate representation of F is given by

$$\widehat{F}(x_1, \dots, x_r, x_{r+1}, \dots, x_m) = (x_1, \dots, x_r, 0, \dots, 0).$$

Before proving this, we should mention a couple of things:

- If $m = n = r$, then this follows immediately from the inverse function theorem.
- The global condition on the rank of F cannot be weakened, as the space of $n \times m$ matrices of rank r need *not* be open. For example, consider the map $A(t) \equiv \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$, which has rank 2 when $t \neq 1$ and rank 1 otherwise.

Proof. Since our statement is local, we may assume that $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are open subsets. Since $DF(p)$ has rank r , it has some invertible $r \times r$ sub-matrix, which we may assume is the upper left sub-matrix $\left(\frac{\partial F^i}{\partial x^j}\right)_{i,j \in [r]}$. Write $(x, y) = (x^1, \dots, x^r, y^1, \dots, y^{m-r})$ and $(v, w) = (v^1, \dots, v^r, w^1, \dots, w^{n-r})$ for the standard coordinates on \mathbb{R}^m and \mathbb{R}^n , respectively. By applying suitable translations, we may assume that $p = (0, 0)$ and $F(p) = (0, 0)$. We have $F(x, y) = (Q(x, y), R(x, y))$ for some smooth map $Q : M \rightarrow \mathbb{R}^r$ and $R : M \rightarrow \mathbb{R}^{n-r}$. Then the Jacobian matrix $\left(\frac{\partial Q^i}{\partial x^j}\right)$ is invertible at $(0, 0)$ by hypothesis.

Define $f : M \rightarrow \mathbb{R}^m$ by $(x, y) \mapsto (Q(x, y), y)$. Define the *Kronecker delta* symbol δ_i^j by

$$\delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Then

$$D[f](0, 0) \begin{bmatrix} \frac{\partial Q^i}{\partial x^j}(0, 0) & \frac{\partial Q^i}{\partial y^j}(0, 0) \\ 0 & \delta_j^i \end{bmatrix}.$$

Since

$$\det(D[f](0, 0)) = \det\left(\frac{\partial Q^i}{\partial x^j}(0, 0)\right) \cdot \det(\delta_j^i) = \det\left(\frac{\partial Q^i}{\partial x^j}(0, 0)\right) \neq 0,$$

it follows that $D[f]$ is invertible at $(0, 0)$.

Thus, we can apply the inverse function theorem to get a connected open set $U_0 \ni (0, 0)$ and an open cube $\tilde{U}_0 \ni f(0, 0) = (0, 0)$ such that $f : U_0 \rightarrow \tilde{U}_0$ is a diffeomorphism. Let $f^{-1}(x, y) = (A(x, y), B(x, y))$. Then $(x, y) = f(A(x, y), B(x, y)) = (Q(A(x, y), B(x, y)), B(x, y))$, so that $y = B(x, y)$. Hence

$$f^{-1}(x, y) = (A(x, y), y).$$

Additionally, $Q(A(x, y), y) = x$ since $f \circ f^{-1} = \text{id}_{\tilde{U}_0}$. If $\tilde{R} : \tilde{U}_0 \rightarrow \mathbb{R}^{n-r}$ is defined by $(x, y) \mapsto R(A(x, y), y)$, then

$$F \circ f^{-1}(x, y) = (x, \tilde{R}(x, y)).$$

Therefore,

$$D[F \circ f^{-1}](x, y) = \begin{bmatrix} \delta_j^i & 0 \\ \frac{\partial \tilde{R}^i}{\partial x^j}(x, y) & \frac{\partial \tilde{R}^i}{\partial y^j}(x, y) \end{bmatrix}$$

for any $(x, y) \in \tilde{U}_0$. It's clear that the first r columns of this matrix are linearly independent. But since f^{-1} is a diffeomorphism, it has rank r on \tilde{U}_0 . It follows that $\frac{\partial \tilde{R}^i}{\partial y^j}(x, y) = 0$ for each $(x, y) \in \tilde{U}_0$. But \tilde{U}_0 was chosen to be an open cube, so that $\tilde{R}(x, y) = \tilde{R}(x, 0)$. If $S(x) := \tilde{R}(x, 0)$, then $F \circ f^{-1}(x, y) = (x, S(x))$.

Now, let

$$V_0 = \left\{ (v, w) \in N \mid (v, 0) \in \tilde{U}_0 \right\},$$

which is a neighborhood of $(0, 0)$ in N . Since \tilde{U}_0 is a cube, we see that $F \circ f^{-1}(\tilde{U}_0) \subset V_0$. Hence $F(U_0) \subset V_0$. Define $g : V_0 \rightarrow \mathbb{R}^n$ by $(v, w) \mapsto (v, w - S(v))$, which is smooth with inverse $g^{-1}(s, t) = (s, t + S(s))$. Then

$$\hat{F}(x, y) = g \circ F \circ f^{-1}(x, y) = (x, S(x) - S(x)) = (x, 0),$$

as desired. \square

3.5 Lecture 10

Definition 3.5.1. Consider a smooth map $F : M \rightarrow N$.

1. It is a (*smooth*) *submersion* if it has constant rank equal to $\dim(N)$.
2. It is a (*smooth*) *immersion* if it has constant rank equal to $\dim(M)$.

Definition 3.5.2. A *topological embedding* is a continuous map $F : M \rightarrow N$ which is a homeomorphism onto $F(M)$.

Example 3.5.3.

1. The map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $t \mapsto (t^3, 0)$ is a smooth topological embedding but not an immersion, since $\gamma'(0) = 0$.
2. The curve $f : (-\pi, \pi) \rightarrow \mathbb{R}^2$ defined by $f(t) = (\sin 2t, \sin t)$ is known as a *lemniscate*, describing a figure-eight curve. This is not a topological embedding, because the figure-eight is compact whereas $(-\pi, \pi)$ is not. But it is a smooth immersion as f' never vanishes.

Definition 3.5.4. A map is a *smooth embedding* if it both a topological embedding and a smooth immersion.

Example 3.5.5.

1. There is a smooth embedding of \mathbb{RP}^2 into \mathbb{R}^4 but not into \mathbb{R}^3
2. If $U \subset M$ is open, then the inclusion $U \hookrightarrow M$ is a smooth embedding.

Definition 3.5.6. A manifold $S \subset M$ in the subspace topology is an *embedded submanifold* if it has a smooth structure such that the inclusion $S \hookrightarrow M$ is a smooth embedding.

Note 3.5.7. The image of a smooth embedding is an embedded submanifold.

Terminology. If $S \subset M$ is an embedded submanifold, then $\dim(M) - \dim(S)$ is called the *codimension of S in M* .

Proposition 3.5.8. Let $U \subset M^m$ be open and $f : U \rightarrow N$ be smooth. The graph $\Gamma(f)$ of f is an embedded m -dimensional submanifold of $M \times N$.

Proof. Define $\gamma_f(x) : U \rightarrow M \times N$ by $\gamma_f(x) = (x, f(x))$. It's easy to check this is a smooth embedding. \square

Our next notion is a local version of the standard embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$ where $k \leq n$ but works for any submanifold.

Definition 3.5.9. We say that a subset $S \subset M$ has the *local k -slice condition* if for each $p \in S$, there is a chart $(U, \varphi) \ni p$ for M such that

$$\varphi(U \cap S) = \underbrace{\{x \in \varphi(U) : x^{k+1} = \dots = x^n = 0\}}_{k\text{-slice of } \varphi(U)}, \quad n \equiv \dim(M).$$

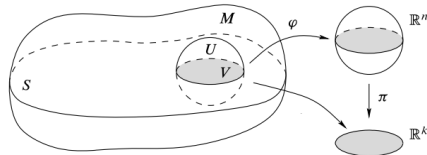


Figure 1: Copied from Lee (102)
 k -slice condition with $V \equiv U \cap S$

Theorem 3.5.10. Let M^n be a smooth manifold. Let $S \subset M$. If S is an embedded manifold with $\dim(S) = k$, then S has the local k -slice condition.

Conversely, if S has the local k -slice condition, then S is a smooth manifold in the subspace topology and has a smooth structure making it an embedded submanifold of dimension k .

Proof.

(\implies)

Let $p \in S$. In particular, the inclusion $i : S \hookrightarrow M$ is a smooth immersion and thus has constant rank k . By the constant rank theorem, we can find charts (U, φ) and (V, ψ) centered at p for S and M , respectively, for which i has coordinate representation

$$(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0).$$

This means that $i(U)$ is a k -slice for S in V . We have that $U = W \cap S$ for some open set W in M . Let $V' = W \cap V$, which is neighborhood of p in M . Then $(V', \psi|_{V'})$ is a chart on M such that $V' \cap S = i(U)$, so that V' is slice for S in M .

(\impliedby)

See Theorem 5.8 (Lee). □

Example 3.5.11. For any n , $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is an embedded hypersurface because it is locally the graph of smooth map and thus has the local n -slice condition.

Theorem 3.5.12. Let $F : M^m \rightarrow N^n$ be smooth with constant rank r . Each level set of F is an embedded submanifold of codimension r in M .

Proof. Set $k = m - r$. Let $c \in N$ and $p \in F^{-1}(c)$. By the constant rank theorem, there are charts (U, f) centered at p and (V, g) centered at $F(p) = c$ for which F has coordinate representation given by

$$(x_1, \dots, x_r, x_{r+1}, \dots, x_m) \mapsto (x_1, \dots, x_r, 0, \dots, 0),$$

which must send each point in $f(F^{-1}(c) \cap U)$ to 0. Thus, $f(F^{-1}(c) \cap U)$ equals the k -slice

$$\{x \in \mathbb{R}^m : x_1 = \cdots = x_r = 0\}.$$

By Theorem 3.5.10, S is an embedded submanifold of dimension k . □

3.6 Lecture 11

Question. Can M^n with $n \geq 1$ be homeo-/diffeomorphic to $M \setminus \{p\}$?

Remark 3.6.1. We can generalize Theorem 3.5.12 to maps that are not necessarily of constant rank.

Definition 3.6.2. Let $\varphi : M \rightarrow N$ be smooth. We say that $p \in M$ is

- a *regular point* of φ if $d\varphi_p$ is surjective and
- a *critical point* of φ otherwise.

Definition 3.6.3. Let $\varphi : M \rightarrow N$ be smooth. We say that $c \in N$ is

- a *regular value* of φ if each point in $\varphi^{-1}(c)$ is regular and
- a *critical value* of φ otherwise.

We say that $S \subset M$ is a *regular level set* of φ if it has the form $\varphi^{-1}(c)$ with c a regular value.

Theorem 3.6.4. *Every regular level set S of a smooth map $F : M^m \rightarrow N^n$ is an embedded submanifold of codimension n .*

Proof. Let $S = F^{-1}(c)$. Note that the subspace of full-rank matrices is open due to continuity of the det. As a result, the set U of points $p \in M$ where dF_p is surjective is open in M . Hence $F|_U : U \rightarrow N$ is a smooth submersion. In particular, it has constant rank n . Thanks to Theorem 3.5.12, it follows that $F^{-1}(c)$ is an embedded submanifold of U with codimension n , where U itself is an open submanifold of M . □

Example 3.6.5. \mathbb{S}^n is a regular level set of the smooth function $\vec{x} \mapsto |\vec{x}|^2$.

Theorem 3.6.6 (Sard). *If $F : M \rightarrow N$ is smooth, then the set of all critical values of F has measure zero in N .*

Proposition 3.6.7. *Suppose M is smooth and $S \subset M$ is embedded. Then for any $f \in C^\infty(S)$, there is some neighborhood U of S in M along with some $\hat{f} \in C^\infty(U)$ such that $\hat{f}|_S = f$.*

Proposition 3.6.8. *The tangent space of a submanifold $S \subset M$ at $p \in S$ is precisely the image of the injective canonical map $di_p : T_p S \rightarrow T_p M$ where i denotes inclusion, i.e.,*

$$A := \{\gamma'(0) \in T_p M : \gamma : (-\epsilon, \epsilon) \rightarrow S \text{ and } \gamma(0) = p\}.$$

Proof. Let $v \in T_p S$. We know that $v = \gamma'(0)$ for some curve γ in S . Then $i \circ \gamma$ is a curve in M with $(i \circ \gamma)' = di_p(v)$.

Conversely, let $v := w'(0) \in A$. We have $w = j \circ w$ where $j : i(S) \rightarrow S$ is the reverse inclusion. Since $(j \circ w)'(0) = dj_p(v) \in T_p S$, it follows that $d_i((j \circ w)'(0)) = v$. □

At this point, we begin developing the theory of differential forms. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be smooth. The gradient ∇F has two main properties.

1. It is orthogonal to the level sets of F .
2. $dF_p(v) = \langle \nabla F_p, v \rangle$.

But given a smooth manifold M , we don't necessarily have an inner product on M unless M is a *Riemannian manifold*, which by definition has a smoothly varying inner product. Instead, we shall view dF_p as a so-called 1-form.

3.7 Lecture 12

Recall that if $\pi : M \rightarrow N$ is a continuous map, then a *section of π* is a continuous right inverse of π .

Definition 3.7.1. A (*smooth*) *vector field* X is a smooth section of the projection map $\pi : TM \rightarrow M$, i.e., $X_p := F(p) \in T_p M$ for each $p \in M$.

Notation. Let $\mathcal{X}(M)$ denote the vector space of all smooth vector fields in M .

Note that $\mathcal{X}(M)$ is a module over $C^\infty(M)$ under the action $f \cdot X \equiv (p \mapsto f(p)X_p)$.

Given a chart U on M^n , if $p \in U$, then we can write $X_p = \sum_{i=1}^n r_i \frac{\partial}{\partial x_i} \Big|_p$ for some unique real coefficients r_i . Define $X^i : U \rightarrow \mathbb{R}$ by $X_i(p) = r_i$ for each $i = 1, \dots, n$. Then

$$X_p = \sum_i X_i(p) \frac{\partial}{\partial x_i} \Big|_p.$$

We call such X_i the *component functions of X* for the chart U .

Proposition 3.7.2. A vector field X is smooth if and only if each component function in any given chart is smooth.

Lemma 3.7.3. If S is a closed subset of M and X a smooth vector field along S , then there is an extension of X to a smooth vector field on M .

Definition 3.7.4. Let $U \subset M^n$ be open and $X_1, \dots, X_k \in \mathcal{X}(M)$.

1. X_1, \dots, X_k are *linearly independent* if for any $p \in U$, we have that $\{X_1(p), \dots, X_k(p)\}$ is linearly independent in $T_p M$.
2. If $k = n$ and X_1, \dots, X_k are linearly independent, then $\{X_1, \dots, X_k\}$ is a *local frame* in U .

Example 3.7.5. The basis vectors $p \mapsto \frac{\partial}{\partial x_i} \Big|_p$ form a local frame for a given chart U around p , called the *coordinate frame*.

Definition 3.7.6. A local frame for U is called a *global frame* if $U = M$. If such a frame exists, then M is called *parallelizable*.

Example 3.7.7. \mathbb{R}^n is parallelizable via the standard coordinate vector fields.

Lemma 3.7.8. M is parallelizable if and only if $TM \approx M \times \mathbb{R}^n$, i.e., its tangent bundle is trivial.

Theorem 3.7.9 (Kervaire). \mathbb{S}^n is parallelizable if and only if $n \in \{0, 1, 3, 7\}$.

Definition 3.7.10 (Lie group). A Lie group is a group G equipped with a smooth structure such that both $\times : G \times G \rightarrow G$ and $(-)^{-1} : G \rightarrow G$ are smooth maps.

Example 3.7.11. Any Lie group is parallelizable.

Note that $\mathcal{X}(M)$ acts on $C^\infty(U)$ for any $U \subset M$ with the action $X \cdot f \equiv (p \mapsto X_p(f))$. Given $X \in \mathcal{X}(M)$, this induces a linear map $X : C^\infty(U) \rightarrow C^\infty(U)$ satisfying the product rule

$$X(fg) = fXg + gXf.$$

We call such a map a *derivation* of $C^\infty(U)$.

Moreover, if $F : M \rightarrow N$ is smooth, then $dF_p X(p) \in T_{F(p)}N$ for each $p \in M$. Yet, this may *not* define a vector field on N , since F may not be surjective.

Example 3.7.12. Let $X, Y \in \mathcal{X}(M)$. Then $X(Yf)$ need *not* be a derivation. Indeed, let $M = \mathbb{R}^2$, $X = \frac{\partial}{\partial x}$, and $Y = x \frac{\partial}{\partial y}$. If $f(x, y) = x$ and $g(x, y) = y$, then $XY(fg) = 2x$ whereas $fXY(g) + gXY(f) = x$, so that $XY(f)$ is not a derivation.

Definition 3.7.13. Let $X, Y \in \mathcal{X}(M)$. The *Lie bracket* of X and Y is

$$[X, Y] \equiv XY - YX : C^\infty(M) \rightarrow C^\infty(M).$$

Proposition 3.7.14 (Clairaut). If $X_i = \frac{\partial}{\partial x_i} \in \mathcal{X}(M)$, then $[X_i, X_j] = 0$ for any $1 \leq i, j \leq n$.

Lemma 3.7.15. A map $D : C^\infty(M) \rightarrow C^\infty(M)$ is a derivation if and only if there is some $X \in \mathcal{X}(M)$ such that $Df = Xf$ for any f .

Proof. We have established the (\Leftarrow) direction. Conversely, assume that D is a derivation. Define $X : M \rightarrow TM$ by $X_p(f) = (Df)(p)$. Since $Df = Xf$ is smooth for each X , it follows that X is smooth thanks to Proposition 8.14 (Lee). \square

Lemma 3.7.16. Any Lie bracket $[X, Y]$ is a smooth vector field.

Proof. By Lemma 3.7.15, it suffices to show that $[X, Y]$ is a derivation. Let f, g be smooth functions on M . Then

$$\begin{aligned} [X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\ &= X(fYg + gYf) - Y(fXg + gXf) \\ &= XfYg + XgYf - YfXg - YgXf \\ &= fXYg + YgXf + gXYf + YfXg \\ &\quad - fYXg - XgYf - gYXf - XfYg \\ &= fXYg + gXYf - fYXg - gYXf \\ &= f[X, Y]g + g[X, Y]f. \end{aligned}$$

\square

3.8 Lecture 13

Consider two smooth vector fields X and Y on M . Define $[X, Y] : M \rightarrow TM$ by $p \mapsto (f \mapsto X_p(Yf) - Y_p(Xf))$.

Proposition 3.8.1. Write $X = X^i \frac{\partial}{\partial x_i}$ and $Y = Y^j \frac{\partial}{\partial x_j}$ in local coordinates. Then

$$[X, Y] = \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x_i} - Y^j \frac{\partial X^i}{\partial x_j} \right) \frac{\partial}{\partial x_j}.$$

Proof. Since $[X, Y]$ is a vector field, we see that $([X, Y]f) \upharpoonright_U = [X, Y](f \upharpoonright_U)$ for any open subset $U \subset M$. Therefore, we may compute, say, Xf in a local coordinate expression for X . To this end, let us apply the product rule together with Clairaut's theorem to get

$$\begin{aligned} [X, Y]f &= X^i \frac{\partial}{\partial x_i} \left(Y^j \frac{\partial f}{\partial x_j} \right) - Y^j \frac{\partial}{\partial x_j} \left(X^i \frac{\partial f}{\partial x_i} \right) \\ &= X^i \frac{\partial Y^j}{\partial x_i} \frac{\partial f}{\partial x_j} + X^i Y^j \frac{\partial^2 f}{\partial x_i \partial x_j} - Y^j \frac{\partial X^i}{\partial x_j} \frac{\partial f}{\partial x_i} - Y^j X^i \frac{\partial^2 f}{\partial x_j \partial x_i} \\ &= X^i \frac{\partial Y^j}{\partial x_i} \frac{\partial f}{\partial x_j} - Y^j \frac{\partial X^i}{\partial x_j} \frac{\partial f}{\partial x_i} \\ &= \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x_i} - Y^j \frac{\partial X^i}{\partial x_j} \right) \frac{\partial}{\partial x_j}. \end{aligned}$$

□

Remark 3.8.2. If $X_1, \dots, X_n \in \mathcal{X}(U)$ satisfy $[X_i, X_j] = 0$, then there are local coordinates $x^i : V \rightarrow \mathbb{R}$ such that $X_i = \frac{\partial}{\partial x^i}$. This is a converse of Clairaut's theorem.

Proposition 3.8.3.

1. (Bilinearity) For any $a, b \in \mathbb{R}$,

$$\begin{aligned} [aX + bY, Z] &= a[X, Z] + b[Y, Z] \\ [Z, aX + bY] &= a[Z, X] + b[Z, Y]. \end{aligned}$$

2. (Antisymmetry)

$$[X, Y] = -[Y, X].$$

3. (Jacobi Identity)

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

4. For any $f, g \in C^\infty(M)$,

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X,$$

where fX denotes the module action $f \cdot X$.

Now, let $X \in \mathcal{X}(M)$ and $Y \in \mathcal{X}(N)$. Let $F : M \rightarrow N$ be a diffeomorphism. The pushforward of X by F , denoted by F_*X , is the vector field on N given by

$$q \mapsto dF_{F^{-1}(q)} (X_{F^{-1}(q)}).$$

We say X and Y are F -related if $Y = F_*X$.

Note 3.8.4. $X(f \circ F) = (Yf) \circ F$ if and only if X and Y are F -related.

Theorem 3.8.5 (Naturality of the Lie bracket). $F_*[X, Y] = [F_*X, F_*Y]$.

Proof. Let $f \in C^\infty(M)$. By Note 3.8.4, we see that $XY(f \circ F) = X(F_*Yf \circ F) = F_*X(F_*Yf) \circ F$, and likewise $YX(f \circ F) = F_*Y(F_*Xf) \circ F$. Thus,

$$[X, Y](f \circ F) = F_*X(F_*Yf) \circ F - F_*Y(F_*Xf) \circ F = ([F_*X, F_*Y]f) \circ F.$$

We conclude by again applying Note 3.8.4. □

Corollary 3.8.6. Let $S \subset M$ be a submanifold. If $X, Y \in \mathcal{X}(M)$ satisfy $X_p, Y_p \in T_p(S)$ for each $p \in S$, then $[X, Y]_p \in T_p(S)$ as well.

Proof. Let $i : S \rightarrow M$ denote inclusion. Then there are $X', Y' \in \mathcal{X}(S)$ with X' i -related to $X \upharpoonright_S$ and Y' i -related to $Y \upharpoonright_S$. This implies that $[X', Y']$ is i -related to $[X, Y] \upharpoonright_S$, which in turn implies that $[X, Y]_p \in T_p(S)$ for any $p \in S$. □

4 Vector bundles

Definition 4.0.1. Let M be a space. A (real) vector bundle of rank k over M is a space E endowed with the following structure.

- (I) A surjective continuous map $\pi : E \rightarrow M$.
- (II) For each $p \in M$, $E_p := \pi^{-1}(p)$ is a k -dimensional vector space.
- (III) For each $p \in M$, there is a neighborhood U_p in M together with a homeomorphism $\varphi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ (called a *local trivialization*) such that
 - (a) $\pi_U \circ \varphi = \pi \upharpoonright_{\pi^{-1}(U)}$, where $\pi_U : U \times \mathbb{R}^k \rightarrow U$ denotes the projection and
 - (b) for each $q \in U$, $\varphi \upharpoonright_{E_q}$ is a linear isomorphism $E_q \xrightarrow{\cong} \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

If M and E are smooth manifolds and each local trivialization is smooth, then E is called a *smooth vector bundle*.

Example 4.0.2. The Möbius strip and $\mathbb{S}^1 \times \mathbb{R}$ are distinct vector bundles of rank 1 over \mathbb{S}^1 .

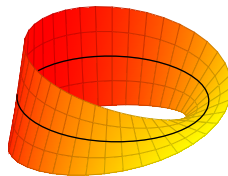


Figure 2: Möbius strip

We can always construct a global section of a smooth vector bundle by using partitions of unity. But we cannot always ensure that it is non-vanishing, as shown by the hairy ball theorem (Corollary 7.3.6) for bundles over \mathbb{S}^2 .

4.1 Lecture 14

Lemma 4.1.1 (Vector bundle construction). *Let M^n be a smooth manifold and suppose that for any $p \in M$, there is some vector space E_p of dimension k . Let $E := \coprod_{p \in M} E_p$ and $\pi : E \rightarrow M$ be the projection map. Further, suppose we have the following data:*

- (a) *an open cover $\{U_\alpha\}$,*
- (b) *for each α , a bijection $\varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a linear isomorphism to $\{p\} \times \mathbb{R}^k$, and*
- (c) *for each $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(k, \mathbb{R})$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v)$.*

Then E has a unique topology and smooth structure making it into a smooth vector bundle of rank k over M .

The matrices $\tau_{\alpha\beta}(p)$ are called the *transition functions* of the vector bundle E . They satisfy the so-called cocycle condition:

$$\tau_{\alpha\alpha}(p) = I_k \quad \tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p)\tau_{\gamma\alpha}(p) = I_k.$$

Definition 4.1.2 (Bundle map). Let $p_1 : E_1 \rightarrow M_1$ and $p_2 : E_2 \rightarrow M_2$ be two vector bundles of rank k . A *homomorphism* $p_1 \rightarrow p_2$ is a commutative square

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ M_1 & \xrightarrow{g} & M_2 \end{array}$$

in the category of spaces such that each map $f|_{p_1^{-1}(x)}$ is linear.

Note that g is uniquely determined by f because p_1 is surjective.

Let us now explore a specific kind of vector bundle. To this end, consider any vector space V as well as its *dual space*

$$V^* \equiv \text{Hom}(V, \mathbb{R}),$$

which consists of all linear maps $V \rightarrow \mathbb{R}$, known as *covectors on V* . If $A : V \rightarrow W$ is linear, then let A^* denote the linear map $W^* \rightarrow V^*$ defined by $w \mapsto (v \mapsto w(Av))$, called the *dual map* of A .

Let $\{v_1, \dots, v_n\}$ be a basis for V . The *dual basis* (or *cobasis*) consists of those linear functionals $\varphi_i : V \rightarrow \mathbb{R}$ given by

$$\varphi_i(v_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}.$$

for each $i = 1, \dots, n$.

Proposition 4.1.3.

- (1) *If $\dim(V) = n$, then $\dim(V^*) = n$.*

Proof. Pick a basis b_1, \dots, b_n for V . Consider its dual basis $\{b^1, \dots, b^n\}$. It is easy to check that this is linearly independent. Further, for any $T \in V^*$, we see that

$$T = T_1 b^1 + \dots + T_n b^n, \quad T_i \equiv T(b_i).$$

This means that the b^i span $\text{Hom}(V, \mathbb{R})$ as well. \square

Remark 4.1.4. The induced isomorphism $V \rightarrow V^*$ is *not* unique, for it depends on our chosen basis of V .

(2) The mapping $v \mapsto \underbrace{(\varphi \mapsto \varphi(v))}_{\text{ev}_v}$ defines a canonical isomorphism

$$V \xrightarrow{\cong} (V^*)^* = \text{Hom}(V^*, \mathbb{R}).$$

Definition 4.1.5. Let M^n be a smooth manifold.

1. Define the *cotangent space at p* as T_p^*M .
2. Define the *cotangent bundle of M* as $T^*M \equiv \coprod_p T_p^*M$.

Lemma 4.1.6. T^*M is a smooth n -vector bundle over M .

Proof. Let (U, φ) be a smooth chart on M . Define $\varphi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ by $a_i \lambda^i|_p \mapsto (p, a_1, \dots, a_n)$ where $\{\lambda^i|_p\}$ is a chosen dual basis for T_pM . Now we apply the vector bundle construction lemma. See Proposition 11.9 (Lee). \square

Let (U, x^i) be smooth coordinates for M^n . Then the map $\psi : a_i \lambda^i|_p \mapsto (x^1(p), \dots, x^n(p), a_1, \dots, a_n)$ makes $(\pi^{-1}(U), \psi)$ a chart on T^*M .

A smooth section of T^*M is called a *covector field* (or *differential/smooth 1-form*) on M . The vector space of such sections will be denoted by $\Gamma(T^*M)$.

Moreover, if U is a chart on M , then a tuple $(\epsilon^1, \dots, \epsilon^k)$ of covector fields on M is a *local coframe* if $\{\epsilon^1|_p, \dots, \epsilon^k|_p\}$ is a basis of T_p^*U for each $p \in U$.

Aside. Let $\pi : E \rightarrow M$ be a smooth vector bundle. The *jet bundle* $J^k E \rightarrow M$ of order k is the smooth vector bundle whose fiber at $p \in M$ consists of all *order- k jets* of smooth sections of π , i.e., equivalence classes of smooth sections of π where two sections are declared equivalent if their first k partial derivatives agree on a neighborhood of p . Note that a germ is precisely an order-1 jet.

We have a sequence of maps

$$\dots J^3 E \rightarrow J^2 E \rightarrow J^1 E \rightarrow E,$$

whose limit is called the *infinite jet bundle* $J^\infty E$.

5 Differential forms

5.1 Lecture 15

Definition 5.1.1 (Differential of a smooth function). Define $C^\infty(M) \rightarrow \Gamma(T^*M)$ by $f \mapsto (p \mapsto df_p)$ where

$$df_p(v) \equiv v f$$

for every $v \in T_p M$. We call df the *differential of f* .

Let (U, x^i) be local coordinates for M . Let (dx^i) denote the corresponding coordinate coframe. We have $df_p = A_i(p)dx^i|_p$ for some functions $A_i : U \rightarrow \mathbb{R}$. Then

$$\begin{aligned} A_i(p) &= df_p \left(\frac{\partial}{\partial x^i} \Big|_p \right) = \frac{\partial f}{\partial x^i}(p) \\ &\quad \downarrow \\ df_p &= \frac{\partial f}{\partial x^i}(p) dx^i|_p. \end{aligned}$$

In this way, the differential of f generalizes the gradient of a smooth function on \mathbb{R}^n .

Proposition 5.1.2. *If M is connected, then f is constant if and only if $df = 0$.*

Proof. Since $v f = 0$ for any derivation v and constant function f , the forward direction is clear. Conversely, suppose that $df = 0$ and let $p \in M$. Set $C = \{q \in M : f(q) = f(p)\}$. We must show that $C = M$. Provided that M is connected, it suffices to show that C is clopen. For any $q \in C$, choose a coordinate ball $U \ni p$. Then since $0 = df = \frac{\partial f}{\partial x^i} dx^i$, it follows that $\frac{\partial f}{\partial x^i} = 0$ for each i . Elementary calculus reveals that f must be constant on U . Hence C is open. Since $C = f^{-1}(f(p))$, it is also closed. \square

Note 5.1.3 (Transition functions for changing coordinates). Let $p \in M$ and suppose that $(x^i)_{1 \leq i \leq n}$ and $(y^i)_{1 \leq i \leq n}$ are two coordinate charts around p . The chain rule for partial derivatives states that

$$\frac{\partial}{\partial x^j} \Big|_p = \sum_k \frac{\partial y^k}{\partial x^j}(\hat{p}) \frac{\partial}{\partial y^k} \Big|_p$$

where $\hat{p} := (x^1(p), \dots, x^n(p))$. Dually, for each $i \in \{1, \dots, n\}$, we have that

$$dx^i|_p = \sum_\ell A_\ell^i dy^\ell|_p$$

for some $A_\ell^i \in \mathbb{R}$, $l = 1, \dots, n$. It follows that

$$\begin{aligned}
\delta_i^j &= dx^i|_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) \\
&= dx^i|_p \left(\sum_k \frac{\partial y^k}{\partial x^j} \frac{\partial}{\partial y^k} \Big|_p \right) \\
&= \sum_k \frac{\partial y^k}{\partial x^j} dx^i|_p \left(\frac{\partial}{\partial y^k} \Big|_p \right) \\
&= \sum_k \frac{\partial y^k}{\partial x^j} \sum_\ell A_\ell^i dy^\ell|_p \left(\frac{\partial}{\partial y^k} \Big|_p \right) \\
&= \sum_k \frac{\partial y^k}{\partial x^j} \sum_\ell A_\ell^i \delta_\ell^k \\
&= \sum_k A_k^i \frac{\partial y^k}{\partial x^j}.
\end{aligned}$$

Therefore, if A denotes the $n \times n$ matrix (A_ℓ^i) and J denotes the Jacobian of (y^1, \dots, y^n) at \hat{p} , then $I_n = JA$, i.e., $A = J^{-1}$.

Definition 5.1.4. Let $F : M \rightarrow N$ be smooth. Let $\omega \in \Gamma(T^*N)$. Define the *pullback* $F^*\omega$ of ω as the element of $\Gamma(T^*M)$ given by

$$F^*\omega|_p \left(X|_p \right) \equiv \omega|_{F(p)} \left(F_*|_p X_p \right).$$

Note that, unlike the pushforward, the pullback requires merely that F be smooth.

Lemma 5.1.5. Let $F : M \rightarrow N$ be smooth, $\alpha, \beta \in \Gamma(T^*N)$ and $f, g \in C^\infty(N)$. Then

$$F^*(f\alpha + g\beta) = (f \circ F) F^*\alpha + (g \circ F) F^*\beta.$$

Proof. Let $X \in \mathcal{X}(M)$. We have that

$$\begin{aligned}
F^*(f\alpha + g\beta)|_p(X_p) &= (f\alpha + g\beta)|_{F(p)} \left(F_*|_p X_p \right) \\
&= f(F(p)) \alpha_{F(p)} \left(F_*|_p X_p \right) + g(F(p)) \beta_{F(p)} \left(F_*|_p X_p \right) \\
&= [(f \circ F) F^*\alpha]|_p(X_p) + [(g \circ F) F^*\beta]|_p(X_p).
\end{aligned}$$

□

Let $\gamma : J \subset \mathbb{R} \rightarrow M$ be a smooth curve in M . Note that $\Gamma(T^*\mathbb{R}) = \{f(t)dt \mid f : T \rightarrow \mathbb{R}\}$. Then

$$\omega \in \Gamma(T^*M) \implies \gamma^*\omega \in \Gamma(T^*\mathbb{R}) \implies \gamma^*\omega = f(t)dt$$

for some curve f along J . This enables us to modestly generalize our notion of integration.

Definition 5.1.6. The *integral of ω along γ* is

$$\int_\gamma \omega \equiv \int_J \gamma^*\omega.$$

Proposition 5.1.7. *Suppose that φ is a positive reparameterization of γ (i.e., one with positive derivative). Then $\int_\gamma \omega = \int_{\gamma \circ \varphi} \omega$.⁴*

Definition 5.1.8. A differential 1-form ω on a smooth manifold M is *closed* if the equation

$$\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = 0$$

holds for any i, j in any chart on M .

Exercise 5.1.9. *Show that being closed is a well-defined property.*

Example 5.1.10. By Clairaut's theorem, df is closed for any $f \in C^\infty(M)$.

5.2 Lecture 16

Recall that a map $T : V_1 \times \cdots \times V_k \rightarrow W$ of vector spaces is *multilinear* if it is linear in each argument, i.e.,

$$T(v_1, \dots, ax + by, \dots, v_k) = aT(v_1, \dots, x, \dots, v_k) + bT(v_1, \dots, y, \dots, v_k)$$

for any $a, b \in \mathbb{R}$.

Theorem 5.2.1 (Universal property of the tensor product). *Let V_1, \dots, V_k be vector spaces. There exists a vector space $V_1 \otimes \cdots \otimes V_k$ together with a map $\otimes : V_1 \times \cdots \times V_k$ so that for any multilinear map $T : V_1 \times \cdots \times V_k \rightarrow W$, there is some unique linear map $\tilde{T} : V_1 \otimes \cdots \otimes V_k \rightarrow W$ such that*

$$\begin{array}{ccc} V_1 \times \cdots \times V_k & \xrightarrow{T} & W \\ \otimes \downarrow & \nearrow \tilde{T} & \\ V_1 \otimes \cdots \otimes V_k & & \end{array}$$

commutes.

Terminology. $V_1 \otimes \cdots \otimes V_k$ is called the *tensor product of the V_i* .

Proof. Let us just prove this when $k = 2$, for then we're done by induction. Let $\mathbb{R}\langle V_1 \times V_2 \rangle$ denote the free vector space on $V_1 \times V_2$, which consists of all finite formal linear combinations of $V_1 \times V_2$. Let

$$G = \langle (av_1, v_2) - a(v_1, v_2), (v_1, av_2) - a(v_1, v_2), (v_1 + w_1, v_2) - (v_1, v_2) - (w_1, v_2), (v_1, w_2 + v_2) - (v_1, w_2) - (v_1, v_2) \rangle.$$

Given a multilinear map $T : V_1 \times V_2 \rightarrow W$, define $\tilde{T} : \mathbb{R}\langle V_1 \times V_2 \rangle \rightarrow W$ by

$$\sum a_{(v_1, v_2)}(v_1, v_2) \mapsto \sum a_{(v_1, v_2)}T(v_1, v_2).$$

Since T is multilinear, $G \subset \ker \tilde{T}$. Therefore, the vector space $V_1 \otimes V_2 := \mathbb{R}\langle V_1 \times V_2 \rangle / G$ fits in a commutative triangle

$$\begin{array}{ccc} \mathbb{R}\langle V_1 \times V_2 \rangle & \xrightarrow{\tilde{T}} & W \\ \pi \downarrow & \nearrow \tilde{\tilde{T}} & \\ V_1 \otimes V_2 & & \end{array}.$$

⁴Proposition 11.31 (Lee).

Thus, if $i : V_1 \times V_2 \rightarrow \mathbb{R}\langle V_1 \times V_2 \rangle$ denotes inclusion, then $\tilde{T} \circ \pi \circ i = \tilde{T} \circ i$, which induces our desired diagram. We see that \tilde{T} is unique because it is uniquely determined by elements of the form

$$v_1 \otimes v_2 := [(v_1, v_2)]$$

under T and every element of $V_1 \otimes V_2$ can be written as some linear combination of such elements. \square

A basic property of the tensor product is that its generic elements are bilinear in the following sense.

Proposition 5.2.2. *If $a, b \in \mathbb{R}$, then $(av_1 + bw_1) \otimes v_2 = a(v_1 \otimes v_2) + b(w_1 \otimes v_2)$.*

Proposition 5.2.3.

1. $(\mathbf{Vect}_{\mathbb{R}}, \oplus, \otimes)$ is a semiring.
2. $V \otimes W \cong W \otimes V$.
3. $V \otimes \mathbb{R} \cong V$.
4. $(V \otimes W)^* \cong V^* \otimes W^*$.

Let $B(V, W)$ denote the space of bilinear maps $V \times W \rightarrow \mathbb{R}$.

Lemma 5.2.4. *There is a canonical isomorphism $V^* \otimes W^* \cong B(V, W)$.*

Proof. Define $\Phi : V^* \times W^* \rightarrow B(V, W)$ by $(\omega, \eta) \mapsto ((v, w) \mapsto \omega(v)\eta(w))$. This is linear and hence induces a commutative diagram

$$\begin{array}{ccc} V^* \times W^* & \xrightarrow{\Phi} & B(V, W) \\ \pi \downarrow & \nearrow \tilde{\Phi} & \\ V^* \otimes W^* & & \end{array} .$$

To see that $\tilde{\Phi}$ is an isomorphism, pick bases $\{f_1, \dots, f_n\}$ and $\{g_1, \dots, g_n\}$ for V and W , respectively. Consider their respective dual bases $\{\xi\}$ and $\{\eta\}$. Then $\{\xi^i \otimes \eta^j : 1 \leq i, j \leq n\}$ is a basis for $V^* \otimes W^*$. Define the linear map $\Psi : B(V, W) \rightarrow V^* \otimes W^*$ by

$$b \mapsto \sum_{i,j} b(f_i, g_j) \xi^i \otimes \eta^j.$$

It is straightforward to check that Ψ is the inverse of $\tilde{\Phi}$. \square

We can generalize Theorem 7.2.3 to obtain an isomorphism

$$V_1^* \otimes \dots \otimes V_k^* \cong L(V_1, \dots, V_k; \mathbb{R}).$$

Definition 5.2.5 (Tensor type). We say that an element of

$$V_\ell^k := \underbrace{V^* \otimes \dots \otimes V^*}_{k \text{ copies}} \otimes \underbrace{V \otimes \dots \otimes V}_{\ell \text{ copies}}$$

is a (k, ℓ) -tensor.

Terminology.

1. A $(k, 0)$ -tensor is called *covariant*.
2. A $(0, \ell)$ -tensor is called *contravariant*.

Let M be a smooth manifold. Define the (k, ℓ) -tensor bundle as

$$T_\ell^k M \equiv \coprod_{p \in M} (T_p)_\ell^k M.$$

In particular, $T^1 M = T^* M$, and $T_1 M = TM$.

Exercise 5.2.6. Find the dimension of $T_\ell^k M$.

Let us examine the form of a generic $(k, 0)$ -tensor. Suppose that (x^j) and (y^i) are two local coordinate systems around a point $p \in M$. Then

$$\begin{aligned} dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k} &= \left(\frac{\partial x^{i_1}}{\partial y^{\ell_1}} dy^{p_1} \right) \otimes \cdots \otimes \left(\frac{\partial x^{i_k}}{\partial y^{\ell_k}} dy^{p_k} \right) \\ &= \sum_{p_1, \dots, p_k} \frac{\partial x^{i_1}}{\partial y^{\ell_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{\ell_k}} dy^{p_1} \otimes \cdots \otimes dy^{p_k}. \end{aligned}$$

Definition 5.2.7. A (k, ℓ) -tensor field is a (smooth) section of $T_\ell^k M$.

Let $\mathcal{T}_\ell^k(M)$ denote the space $\Gamma(T_\ell^k M)$ of all such sections.

5.3 Lecture 17

Let (U, x^i) be local coordinates for M . Then any $A \in \overline{\mathcal{T}_\ell^k(M)}$ can be written in U as

$$A|_p = A_{i_1 \dots i_k}^{j_1 \dots j_\ell} dx^{i_1}|_p \otimes \cdots \otimes dx^{i_k}|_p \otimes \frac{\partial}{\partial x^{j_1}}|_p \otimes \cdots \otimes \frac{\partial}{\partial x^{j_\ell}}|_p,$$

summed over $n^k n^\ell$ many tensors.

Example 5.3.1. Let $\sigma = \delta_j^i dx^j \otimes \frac{\partial}{\partial x^i}$, $X = X^k \frac{\partial}{\partial x^k}$, and $w = w_\ell dx^\ell$. Then

$$\begin{aligned} \sigma(X, w) &= \delta_j^i dx^j \otimes \frac{\partial}{\partial x^i} \left(X^k \frac{\partial}{\partial x^k}, w_\ell dx^\ell \right) \\ &= \delta_j^i dx^j \left(X^k \frac{\partial}{\partial x^k} \right) \frac{\partial}{\partial x^i} w_\ell dx^\ell \\ &= \delta_j^i \delta_k^j X^k w_\ell \delta_i^\ell \\ &= w_k X^k \\ &= w(X). \end{aligned}$$

We say that σ is *invariant* in this case.

Example 5.3.2. Show that the tensor $\delta_i^j dx^i \otimes dx^j$ is *not* invariant.

Proposition 5.3.3.

1. Any $\sigma \in \mathcal{T}_\ell^k(M)$ induces a $C^\infty(M)$ -multilinear map

$$\begin{aligned} \hat{\sigma} : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{k \text{ copies}} \times \underbrace{\mathcal{X}^*(M) \times \cdots \times \mathcal{X}^*(M)}_{\ell \text{ copies}} &\longrightarrow C^\infty(M) \\ (X_1, \dots, X_k, w_1, \dots, w_\ell) &\mapsto \left(p \mapsto \sigma \left(X_1|_p, \dots, X_k|_p, w_1|_p, \dots, w_\ell|_p \right) \right). \end{aligned} \quad (*)$$

2. Any multilinear map over $C^\infty(M)$ is of the form (1) for some (k, ℓ) -tensor field.

Notice that the smooth function $\hat{\sigma}_p$ induced by σ of Example 5.3.1 is determined completely by the values $X_1(p), \dots, X_k(p), w_1(p), \dots, w_\ell(p)$.

Note 5.3.4. The Lie bracket is *not* multilinear over $C^\infty(M)$, for

$$[fX + gY, Z] = f[X, Y] + g[Y, Z] - Z(f)X - Z(g)Y.$$

Definition 5.3.5. A covariant k -tensor T is *alternating* if for any vectors Y, X_1, \dots, X_{k-1} , it follows that

$$T(X_1, X_2, \dots, Y, \dots, Y, \dots, X_{k-1}) = 0.$$

In this case, T is also called an *exterior form*.

Example 5.3.6. If σ is a 0-tensor or a 1-tensor, then it is alternating.

Proposition 5.3.7. *TFAE.*

1. T is alternating.
2. $T(X_1, \dots, X_k) = 0$ whenever $\{X_1, \dots, X_k\}$ is linearly dependent.
3. $T(X_1, \dots, X_i, X_{i+1}, \dots, X_k) = -T(X_1, \dots, X_{i+1}, X_i, \dots, X_k)$.

Notation. The expression $\bigwedge^k(V)$ will denote the subspace of $T^k(V)$ consisting of alternating covariant k -tensors.

Definition 5.3.8. Given $T \in T^k(V)$, the *alternation* $\text{Alt}(T)$ of T is the multilinear map defined by

$$(V_1, \dots, V_k) \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) T(V_{\sigma(1)}, \dots, V_{\sigma(k)}).$$

Example 5.3.9.

1. $\text{Alt}(T)(X, Y) = \frac{1}{2} (T(X, Y) - T(Y, X))$.
2. $\text{Alt}(T)(X, Y, Z) = \frac{1}{6} (T(X, Y, Z) + T(Y, Z, X) + T(Z, X, Y) - T(Y, X, Z) - T(Z, Y, X) - T(X, Z, Y))$.

Example 5.3.10. Suppose that $\{w^1, \dots, w^n\}$ is the cobasis of the standard basis $\{e_1, \dots, e_n\}$ for the vector space V . Then

$$\begin{aligned} &\text{Alt}(w^1 \otimes \cdots \otimes w^n)(e_1, \dots, e_n) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) w^1 \otimes \cdots \otimes w^n(e_{\sigma(1)}, \dots, e_{\sigma(n)}) \\ &= \frac{1}{n!} \text{sgn}(\text{id}_n) w^1 \otimes \cdots \otimes w^n(e_1, \dots, e_n) \\ &= \frac{1}{n!}. \end{aligned}$$

Proposition 5.3.11.

1. $\text{Alt}(T) \in \bigwedge^k(V)$.
2. $\text{Alt}(T) = T \iff T \in \bigwedge^k(V)$.
3. The induced map $\text{Alt} : T^k(V) \rightarrow \bigwedge^k(V)$ is linear.

5.4 Lecture 18

Lemma 5.4.1. Let V be a vector space of dimension $k < \infty$. Let $\{w^1, \dots, w^n\}$ be a cobasis for V . Let $k \leq n$. Then

$$A := \{\text{Alt}(\omega^{i_1} \otimes \dots \otimes \omega^{i_k}) : 1 \leq i_1 < \dots < i_k \leq n\}$$

is a basis for $\bigwedge^k(V)$.

Proof. It's clear from Proposition 5.3.11 that A spans $\bigwedge^k(V)$. It remains to show that A is linearly independent.

Claim.

- (a) If the integers i_1, \dots, i_k are not pairwise distinct, then $\text{Alt}(\omega^{i_1} \otimes \dots \otimes \omega^{i_k}) = 0$.
- (b) $\text{Alt}(\omega^{i_1} \otimes \dots \otimes \omega^{i_j} \otimes \omega^{i_{j+1}} \otimes \dots \otimes \omega^{i_k}) = -\text{Alt}(\omega^{i_1} \otimes \dots \otimes \omega^{i_{j+1}} \otimes \omega^{i_j} \otimes \dots \otimes \omega^{i_k})$.

As a consequence, $\text{span}(A) = \text{span}\{\text{Alt}(\omega^{i_1} \otimes \dots \otimes \omega^{i_k}) : 1 \leq i_1 \leq \dots \leq i_k \leq n\}$.

Exercise 5.4.2. Show that this implies that A is linearly independent. □

Corollary 5.4.3. If $\dim(V) = n$, then $\dim\left(\bigwedge^k(V)\right) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Definition 5.4.4. Define the *wedge product* as the map

$$\wedge : \bigwedge^k(V) \times \bigwedge^\ell(V) \rightarrow \bigwedge^{k+\ell}(V) \quad (w, q) \mapsto w \wedge q \equiv \frac{(k+\ell)!}{k!\ell!} \text{Alt}(w \otimes q).$$

This is like the tensor product.

Example 5.4.5. With notation as in Example 5.3.10, we have that $\omega^1 \wedge \dots \wedge \omega^n(e_1, \dots, e_n) = 1$.

Lemma 5.4.6. The set $\{\omega^{i_1} \wedge \dots \wedge \omega^{i_k} : 1 \leq i_1 < \dots < i_k \leq n\}$ is a basis for $\bigwedge^k(V)$.

Proof sketch. For each k -tuple (i_1, \dots, i_k) , one can show that $\omega^{i_1} \wedge \dots \wedge \omega^{i_k}$ and $\text{Alt}(\omega^{i_1} \otimes \dots \otimes \omega^{i_k})$ differ precisely by a real factor. This is enough thanks to Lemma 5.4.1. □

Consider the standard basis $B := \{e_1, \dots, e_n\}$ for V . Note that $\det \in \bigwedge_B^n(V)$ by Proposition 5.3.11. But $\bigwedge^n(V) = 1$, so that $\det = c(\omega^1 \wedge \dots \wedge \omega^n)$. But evaluating both sides at (e_1, \dots, e_n) yields the equation $1 = c(1) = c$. Thus,

$$\det = \omega^1 \wedge \dots \wedge \omega^n.$$

Proposition 5.4.7. *Suppose that ω , ω' , η , and η' are exterior forms. The following are properties of the wedge product.*

(1) *(Bilinearity) If $a, a' \in \mathbb{R}$, then*

$$\begin{aligned}(a\omega + a'\omega') \wedge \eta &= a(\omega \wedge \eta) + a'(\omega' \wedge \eta) \\ \eta \wedge (a\omega + a'\omega') &= a(\eta \wedge \omega) + a'(\eta \wedge \omega').\end{aligned}$$

(2) *(Associativity)*

$$(\eta \wedge \omega) \wedge \omega' = \eta \wedge (\omega \wedge \omega').$$

(3) *(Anticommutativity) If $\omega \in \wedge^k(V)$ and $\eta \in \wedge^\ell(V)$, then*

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.$$

Corollary 5.4.8. *If ω is a 1-form, then $\omega \wedge \omega = 0$.*

(4) *If $\omega^1, \dots, \omega^k \in \wedge^1(V)$, then*

$$\omega^1 \wedge \dots \wedge \omega^k(v_1, \dots, v_k) = \det(\omega^j(v_i)).$$

Definition 5.4.9. Let M^n be a smooth manifold. Define the *alternating bundle of rank k* as

$$\wedge^k(M) \equiv \coprod_{p \in M} \wedge^k(T_p M).$$

A smooth section of $\wedge^k(M)$ is called a *(differential) k -form*.

Let both $\Omega^k(M)$ and $\mathcal{A}^k(M)$ stand for the infinite-dimensional vector space of differential k -forms on the manifold M . We also have a graded associative algebra $(\Omega^*(M), \wedge)$ over \mathbb{R} .

In local coordinates we have a basis $\left\{ \frac{\partial}{\partial x^i} \Big|_p \right\}_{1 \leq i \leq n}$ for $T_p M$ as well as a corresponding dual basis $\{dx^i\}$. Then for any $\omega \in \wedge^k(M)$, we can write

$$\omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} \omega_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \quad (1)$$

locally at p . Let $I = \{i_1 < \dots < i_k\}$. Since

$$dx^{i_1} \wedge \dots \wedge dx^{i_k} \left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}} \right) = \delta_J^I$$

where $\delta_J^I = 1$ if and only if $I = J$ as sets, it follows that

$$\omega_{i_1, \dots, i_k} = \omega \left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right). \quad (2)$$

We abbreviate (1) by writing

$$\omega = \omega_I dx^I,$$

where we tacitly sum over the I . In this case, for any other ordered set of indices $J := \{j_1 < \dots < j_k\}$, we have

$$\omega \left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}} \right) = \omega_I dx^I \left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}} \right) = \omega_I \delta_J^I.$$

Note 5.4.10. Let $w = w_I dx^I$ and $w = \tilde{w}_J d\tilde{x}^J$ be two coordinate representations of w . Observe that

$$\begin{aligned} \tilde{w}_J &= \omega \left(\frac{\partial}{\partial \tilde{x}^{j_1}}, \dots, \frac{\partial}{\partial \tilde{x}^{j_k}} \right) && ((2)) \\ &= \omega \left(\sum_t \frac{\partial x^{i_t}}{\partial \tilde{x}^{j_1}} \frac{\partial}{\partial x^{i_t}}, \dots, \sum_t \frac{\partial x^{i_t}}{\partial \tilde{x}^{j_k}} \frac{\partial}{\partial x^{i_t}} \right) && (\text{chain rule}) \\ &= \sum_{\sigma \in S_k} \text{sgn}(\sigma) \frac{\partial x^{i_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial \tilde{x}^{j_k}} \omega \left(\frac{\partial}{\partial x^{\sigma(i_1)}}, \dots, \frac{\partial}{\partial x^{\sigma(i_k)}} \right) && (\text{multilinearity of } \omega) \\ &= \det \left(k \times k \text{ minor of } \frac{\partial x}{\partial \tilde{x}} \text{ relative to } i_1, \dots, i_k \text{ and } j_1, \dots, j_k \right). && (\text{Proposition 5.4.7(4)}) \end{aligned}$$

5.5 Lecture 19

The following notion generalizes Definition 5.1.4 to differential forms of arbitrary degree.

Definition 5.5.1 (Pullback). Let $F : M \rightarrow N$ be smooth and $\omega \in \bigwedge^k(N)$. The *pullback* $F^*\omega$ of ω by F is the differential k -form on M given pointwise by

$$F^*\omega|_p(v_1, \dots, v_k) = \omega_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)).$$

Note that $F^*(-)$ is a linear map $\Omega^k(N) \rightarrow \Omega^k(M)$ over \mathbb{R} .

Lemma 5.5.2 (Naturality of the pullback). $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$.

Proof. This is easily seen from Definition 5.5.1 together with Definition 5.4.4. \square

Lemma 5.5.3. *In any local coordinates, we have that*

$$F^* \left(\sum_I \omega_I dy^{i_1} \wedge \cdots \wedge dy^{i_k} \right) = \sum_I (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F).$$

Proof. It is easy to check that $F^*\omega(X_1, \dots, X_k) = \sum_I \omega_I \circ F dy^I(F_*X_1, \dots, F_*X_k)$. Hence it suffices to show that

$$d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F)(X_1, \dots, X_k) = dy^I(F_*X_1, \dots, F_*X_k).$$

For this, it suffices to show that $d(y^i \circ F)(X) = dy^i(F_*X)$ for each $i \in \{i_1, \dots, i_k\}$. Let (x^i) denote local coordinates on M . On the one hand, thanks to Definition 5.1.1, we see that

$$d(y^i \circ F)(X) = X(y^i \circ F) = X^j \frac{\partial F^i}{\partial x^j}.$$

On the other hand, we see that

$$\begin{aligned} dy^i(F_*X) &= dy^i \left(X^j \frac{\partial F^r}{\partial x^j} \frac{\partial}{\partial y^r} \right) \\ &= X^j \frac{\partial F^i}{\partial x^j}. \end{aligned}$$

\square

Example 5.5.4. Consider the change of variables to polar coordinates $\mathbb{R}^2 \rightarrow \mathbb{R}^2$:

$$(r, \theta) \mapsto (r \cos \theta, r \sin \theta).$$

Locally, this is precisely the identity map with the domain endowed with one atlas and the codomain endowed with another. Lemma 5.5.3 together with certain computational properties of \wedge yields

$$\begin{aligned} dx \wedge dy &= d(r \cos \theta) \wedge d(r \sin \theta) \\ &= (\cos \theta dr - r \sin \theta d\theta) \wedge (\sin \theta dr + r \cos \theta d\theta) \\ &= (\cos \theta dr - r \sin \theta d\theta) \wedge \sin \theta dr + (\cos \theta dr - r \sin \theta d\theta) \wedge r \cos \theta d\theta \\ &= (\cos \theta dr \wedge \sin \theta dr) - (r \sin \theta d\theta \wedge \sin \theta dr) + (\cos \theta dr \wedge r \cos \theta d\theta) - (r \sin \theta d\theta \wedge r \cos \theta d\theta) \\ &= -(r \sin \theta d\theta \wedge \sin \theta dr) + (\cos \theta dr \wedge r \cos \theta d\theta) \\ &= r \sin^2 \theta (dr \wedge d\theta) + r \cos^2 \theta (dr \wedge d\theta) \\ &= r dr \wedge d\theta. \end{aligned}$$

Now, let us begin defining a differential operator on smooth forms that generalizes Definition 5.1.1. Let ω be a 1-form on a smooth manifold M . For this to arise as the differential of a smooth function df , each component function ω_i must have the form $\frac{\partial f}{\partial x^i}$. By Clairaut's theorem, this means that ω is closed in the sense of Definition 5.1.8, i.e.,

$$\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = 0 \quad (*)$$

in any chart on M . This property is actually coordinate-independent by Lee (Proposition 11.45). Therefore, we want to express $(*)$ as the ij -component of a 2-form, namely

$$d\omega \equiv \sum_{j < i} \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) dx^j \wedge dx^i.$$

Notice that ω is closed if and only if $d\omega = 0$ in any chart on M .

5.6 Lecture 20

Let $\omega \in \mathcal{A}^k(M)$ with local coordinate representation $\omega_I dx^I$. The *exterior derivative* of ω is the $(k+1)$ -form

$$d\omega \equiv d\omega_I \wedge dx^I.$$

We refer to the operation $d : \mathcal{A}^k(M) \rightarrow \mathcal{A}^{k+1}(M)$ as *exterior differentiation*.

Note 5.6.1. $d\omega = \sum_I \sum_j \frac{\partial}{\partial x^j} \omega_I dx^j \wedge dx^I$.

Aside. If we view $\Omega^k : \mathbf{Diff}^{\text{op}} \rightarrow \mathbf{Vec}_{\mathbb{R}}$ as the functor sending each smooth map f to the pullback f^* , then the exterior derivative becomes a natural transformation $\Omega^k \Rightarrow \Omega^{k+1}$.

Definition 5.6.2. Let $\omega \in \mathcal{A}^k(M)$.

1. We say that ω is *closed* if $d\omega = 0$.
2. We say that ω is *exact* if $\omega = d\eta$ for some $\eta \in \mathcal{A}^{k-1}(M)$.

Lemma 5.6.3. *Suppose that $M = \mathbb{R}^n$, equivalently, that M has a global chart.*

- (1) d is linear over \mathbb{R} .
- (2) $d(F^*\omega) = F^*(d\omega)$.
- (3) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.
- (4) $d \circ d = 0$.

Proof. Statement (1) is obvious. For (2), by linearity, it suffices to consider the case where $\omega = udx^I$. Using Lemma 5.5.3, we compute

$$\begin{aligned} F^*(d(udx^{i_1} \wedge \cdots \wedge dx^{i_k})) &= F^*(du \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\ &= d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F) \\ d(F^*(udx^{i_1} \wedge \cdots \wedge dx^{i_k})) &= d((u \circ F)d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)) \\ &= d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F) \end{aligned}$$

For (3), let $\eta = vdx^J$. Again, by linearity, it suffices to compute $d(udx^I \wedge vdx^J)$.

$$\begin{aligned} d(udx^I \wedge vdx^J) &= d(uvdx^I \wedge dx^J) \\ &= (vdu + udv) \wedge dx^I \wedge dx^J \\ &= (du \wedge dx^I) \wedge (vdx^J) \wedge (dv \wedge udx^I) \wedge dx^J \\ &= (du \wedge dx^I) \wedge (vdx^J) \wedge (-1)^k (udx^I) \wedge (dv \wedge dx^J) \\ &= d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned}$$

To prove (4), first observe that so long as $k = 1$ and $\omega = \omega_j dx^j$, we have that

$$\begin{aligned} d\omega &= \sum_{i,j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i < j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j + \sum_{i > j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i < j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j. \end{aligned}$$

This together with Clairaut's theorem implies that

$$d(du) = \frac{\partial^2 u}{\partial x^i \partial x^j} dx^i \wedge dx^j = \sum_{i < j} \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^i} \right) dx^i \wedge dx^j = 0.$$

Now, drop the assumption that $k = 1$. Then expanding $d(d\omega)$ yields a sum of two summations of wedge products. One of which contains the term $d(d\omega_j)$, and the other contains the term $d(dx^{j_i})$. These both equal zero, and thus the entire expression $d(d\omega)$ vanishes. \square

Corollary 5.6.4 (Naturality of the exterior derivative). *If F is a smooth map, then*

$$d(F^*\omega) = F^*(d\omega).$$

Corollary 5.6.5. *The exterior derivative is well-defined.*

Proof. Let (U, φ) be a chart on M . Notice that

$$d\omega = \varphi^* d(\varphi^{-1*} \omega)$$

on U . Let (V, ψ) be another chart. Then

$$(\varphi \circ \psi^{-1})^* d(\varphi^{-1*} \omega) = d((\varphi \circ \psi^{-1})^* \varphi^{-1*} \omega).$$

Since $(\varphi \circ \psi^{-1})^* = \psi^{-1*} \circ \varphi^*$ and $F^* \circ F^{-1*} = \text{id}$ for any diffeomorphism F , it follows that

$$\begin{aligned} \psi^{-1*} \circ \varphi^* d(\varphi^{-1*} \omega) &= d(\psi^{-1*} \omega). \\ \Downarrow \\ \varphi^* d(\varphi^{-1*} \omega) &= \psi^* d(\psi^{-1*} \omega). \end{aligned}$$

□

Corollary 5.6.6. *Any exact form is closed.*

It is *not* the case, however, that any closed form is exact. Let $M = \mathbb{R}^2 \setminus \{0\}$. Define the 1-form $\omega : M \rightarrow T^*M$ by

$$(x, y) \mapsto \frac{xdy - ydx}{x^2 + y^2}.$$

On the one hand, a straightforward computation shows that $d\omega = 0$. On the other hand, recall from basic calculus that ω is exact on a connected open subset $\omega \subset M$ if and only if $\int_c \omega = 0$ for any closed curve $c \subset \omega$. But if $\gamma : [0, 2\pi] \rightarrow M$ is given by $(\cos \theta, \sin \theta)$, then

$$\int_\gamma \omega = \int_0^{2\pi} d\theta = 2\pi \neq 0, \quad (\dagger)$$

which means that ω is not exact.

Theorem 5.6.7 (Unique differentiation). *The exterior derivative is the unique linear map $\bar{d} : \mathcal{A}^k(M) \rightarrow \mathcal{A}^{k+1}$ such that*

- (i) $\bar{d}(\omega \wedge \eta) = \bar{d}\omega \wedge \eta + (-1)^k \omega \wedge \bar{d}\eta$,
- (ii) $\bar{d}f(X) = Xf$ for any $f \in C^\infty(M)$, and
- (iii) $\bar{d} \circ \bar{d} = 0$.

For example, consider the linear map $\bar{d} : \mathcal{A}^k(M) \rightarrow \mathcal{A}^{k+1}(M)$ given by

$$\begin{aligned} \bar{d}\omega(X_1, \dots, X_{k+1}) &= \sum_{i=1}^{n+1} (-1)^{k+1} X_i \left(w(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) \right) \\ &\quad - \sum_{i,j} (-1)^{i+j} w([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}). \end{aligned}$$

This satisfies conditions (i), (ii), and (iii) of Theorem 5.6.7, and thus $\bar{d} = d$.

To conclude this lecture, let's look at a particular dual operation to exterior differentiation, which will be useful for our discussion of orientation.

Let V be a finite-dimensional vector space. For each vector $v \in V$, define *interior multiplication by v* as the linear map $i_v : \bigwedge^k(V) \rightarrow \bigwedge^{k-1}(V)$ given by

$$i_v \omega(w_1, \dots, w_{k-1}) = \omega(v, w_1, \dots, w_{k-1}).$$

Let $v \lrcorner \omega := i_v \omega$.

Extend interior multiplication as follows. For each $X \in \mathcal{X}(M)$ and $\omega \in \mathcal{A}^k(M)$, define the $(k-1)$ -form $X \lrcorner \omega$ by $p \mapsto X_p \lrcorner \omega_p$.

5.7 Lecture 21

Definition 5.7.1. Let V be a finite-dimensional vector space. Suppose that E and E' are two bases for V . We say that E and E' are *co-oriented* if the change-of-basis matrix from E to E' has positive determinant.

This notion provides us with exactly two equivalence classes of bases for V , which we call the *orientations for V* . If $[E_1, \dots, E_n]$ is a chosen orientation for V , then we call any basis in it (*positively*) *oriented* and any basis not in it *negatively oriented*.

Definition 5.7.2 (Orientation). An *orientation* on a smooth manifold M is a continuous choice of orientation for $T_p M$ as p varies over M .

Equivalently, if $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ denotes the smooth structure on M , we say that M is *orientable* if the Jacobian $D[\varphi_\beta \circ \varphi_\alpha^{-1}]$ has positive determinant on $\varphi_\alpha(U_\alpha \cap U_\beta)$ for any $\alpha, \beta \in A$.

Example 5.7.3. \mathbb{S}^n is orientable for any $n \geq 1$. For each $p \in \mathbb{S}^n$, say that (v_1, \dots, v_n) is positively oriented on $T_p \mathbb{S}^n$ if (p, v_1, \dots, v_n) is positively oriented on \mathbb{R}^{n+1} , i.e., is co-oriented with the standard basis for \mathbb{R}^{n+1} .

Lemma 5.7.4. Let $\pi : E \rightarrow M$ be a smooth vector bundle and $V \subset E$ be open. If V_p is a convex subspace of E_p for every $p \in M$, then there is some $\sigma \in \Gamma(E)$ such that $\sigma_p \in V_p$ for every p .

Proof. Find a cover of E by local trivialisations U_α over M along with smooth sections σ_α of them. There is some partition of unity ψ_α subordinate to (U_α) . Define $\sigma : M \rightarrow E$ as $\sum_\alpha \psi_\alpha \sigma_\alpha$, so that $\sigma \in \Gamma(E)$. Then σ_p belongs to V_p by convexity. \square

Proposition 5.7.5. Suppose that M is a smooth n -manifold. Any nowhere vanishing n -form on M gives rise to a unique orientation on M .

Conversely, any orientation on M gives rise to a nowhere vanishing n -form on M .

Proof.

(\implies)

Let $\omega \in \mathcal{A}^n(M)$ be nowhere vanishing. For each $p \in M$, we see that ω_p defines an orientation O_M^p on M by declaring that $[e_1, \dots, e_n] \in O_M^p$ if and only if $\omega_p(e_1, \dots, e_n) > 0$. It remains to show that if $p \in M$, then we can find some chart U_p around p and some local frame $(E_1, \dots, E_n)_p$ on U_p such that $\omega_q(E_1|_q, \dots, E_n|_q) > 0$

for every $q \in U_p$. To see this, pick any U_p and local frame $(E_1, \dots, E_n)_p$ on U_p . Write $\omega = f dE^1 \wedge \dots \wedge dE^n$ locally for some smooth function $f : U_p \rightarrow \mathbb{R}$. Since ω is nowhere vanishing, it follows that

$$\omega(E_1, \dots, E_n) = f \neq 0.$$

Since f is continuous and M connected, we see that $f > 0$ or $f < 0$. We may assume that $f > 0$ for otherwise we can choose $(-E_1, \dots, -E_n)_p$ instead.

(\Leftarrow)

Given $p \in M$ and an orientation O_M^p on T_pM , say that $w \in \bigwedge^n(T_pM)$ is positively oriented if and only if $w(e_1, \dots, e_n) > 0$ for any $[e_1, \dots, e_n] \in O_M^p$. Then the subspace $\bigwedge_+^n(T_pM)$ is open and convex. By Lemma 5.7.4, we are done. \square

Definition 5.7.6. A diffeomorphism $F : M \rightarrow N$ between two oriented manifolds is *orientation-preserving* if the isomorphism dF_p maps positively oriented bases for T_pM to positively oriented bases for $T_{F(p)}N$ for each $p \in M$. It is *orientation-reversing* if it maps positively oriented bases to negatively oriented ones.

We see that

$$\begin{aligned} F \text{ is orientation-preserving} &\iff \det(dF_p) > 0 \text{ for each } p \in M \\ &\iff F^*\omega \text{ is positively oriented for any positively oriented form } \omega. \end{aligned}$$

Lemma 5.7.7. *The antipodal map $\alpha : \mathbb{S}^n \rightarrow \mathbb{S}^n$ is orientation-preserving if and only if n is odd.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{S}^n & \xrightarrow{\alpha} & \mathbb{S}^n \\ \downarrow & & \downarrow \\ \mathbb{R}^{n+1} & \xrightarrow{\hat{\alpha}} & \mathbb{R}^{n+1} \end{array}$$

where $\hat{\alpha}(\vec{x}) \equiv -\vec{x}$. Note that the Jacobian of $\hat{\alpha}$ is precisely the identity matrix I_{n+1} . Since $\det(I_{n+1}) = (-1)^{n+1}$, we see that $\hat{\alpha}$ is orientation-preserving if and only if n is odd. Thus, the restriction α of $\hat{\alpha}$ to \mathbb{S}^n has the same property. \square

Corollary 5.7.8. $\mathbb{R}\mathbb{P}^n$ is not orientable when n is even.

Proof. Let n be even. Suppose, toward a contradiction, that $\mathbb{R}\mathbb{P}^n$ admits an orientation. Apply Proposition 5.7.5 to obtain a nowhere vanishing n -form ω on $\mathbb{R}\mathbb{P}^n$. If $\pi : \mathbb{S}^n \rightarrow \mathbb{R}\mathbb{P}^n$ denotes the natural projection, then we also obtain the nowhere vanishing n -form $\pi^*\omega$ on \mathbb{S}^n . Applying Proposition 5.7.5 again shows that this determines the usual orientation on \mathbb{S}^n .

Note that $\pi \circ \alpha = \pi$, so that $\alpha^*\pi^*\mathbb{S}^n = \pi^*\mathbb{S}^n$. But this implies that α preserves the orientation of \mathbb{S}^n , contrary to Lemma 5.7.7. \square

The converse of Corollary 5.7.8 is also true, although we omit a proof of it.

Before moving to integration, we should look at a modest variant of our notion of *manifold*. Consider the intersection of \mathbb{R}^n with a half-plane

$$\mathbb{H}^n := \{(x^1, \dots, x^n) \in \mathbb{R}^n : x^n \geq 0\}.$$

Definition 5.7.9 (Manifold with boundary).

1. An n -dimensional manifold with boundary M is a second-countable Hausdorff space that is locally homeomorphic to either an open Euclidean ball or an open subset of \mathbb{H}^n .
2. Any point $p \in M$ is an *interior point* if it belongs to a chart homeomorphic to an open ball.
3. The point p is a *boundary point* if it belongs to a chart that sends p to a point in $\partial\mathbb{H}^n$.

Note that every point in M is either an interior or a boundary point, but not both.

Proposition 5.7.10. *The set of boundary points ∂M is an $(n - 1)$ -dimensional embedded submanifold of M .*

Moreover, ∂M inherits an orientation from M when M is oriented. This is called the *induced* or *Stokes orientation*. Indeed, we may construct a smooth outward-pointing vector field N along ∂M , which is nowhere tangent to ∂M . Therefore, if ω denotes the orientation form for M , then the form $i_{\partial M}^*(N \lrcorner \omega)$ is an orientation form for ∂M .

Example 5.7.11. \mathbb{S}^n is orientable as the boundary of the closed unit ball.

6 Integration

6.1 Lecture 22

Definition 6.1.1. Let $A_0^k(\mathbb{R}^k)$ denote the space of k -forms with compact support. Let $\omega \in A_0^k(\mathbb{R}^k)$ and $\omega = f dx^1 \wedge \cdots \wedge dx^k$. Define

$$\int_{\mathbb{R}^k} \omega = \int_{\mathbb{R}^k} f(x^1, \dots, x^k) dx^1 \cdots dx^k.$$

Exercise 6.1.2. *Given another coordinate representation $\omega = gy^1 \wedge \cdots \wedge y^k$ with $\det\left(\frac{\partial x}{\partial y}\right) > 0$, show that*

$$\int_{\mathbb{R}^k} f(x^1, \dots, x^k) dx^1 \cdots dx^k = \int_{\mathbb{R}^k} g(y^1, \dots, y^k) dy^1 \cdots dy^k.$$

In other words, Definition 6.1.1 makes sense.

A *singular k -cell* on M^n is a smooth map $\sigma : [0, 1]^k \rightarrow M$. Note that 0-cells are precisely points in M and 1-cells are precisely smooth curves in M . Let $\omega \in \mathcal{A}^k(M)$ and σ be a singular k -cell on M . Define

$$\int_{\sigma} \omega = \int_{[0,1]^k} \sigma^* \omega.$$

Proposition 6.1.3. *Let $p : [0, 1]^k \rightarrow [0, 1]^k$ be a diffeomorphism.*

1. *If p is orientation-preserving, then $\int_{\sigma} \omega = \int_{\sigma \circ p} \omega$.*
2. *If p is orientation-reversing, then $\int_{\sigma} \omega = -\int_{\sigma \circ p} \omega$.*

Definition 6.1.4.

1. A *singular k -chain* on M is a formal finite \mathbb{R} -combination $\sigma = \sum_{i=1}^N a_i \sigma_i$ of singular k -cells on M . Define

$$\int_{\sigma} \omega = \sum_{i=1}^N a_i \int_{\sigma_i} \omega.$$

2. Let σ be a singular k -cell on M . Let $i = 1, \dots, 2k$ and $\alpha = 0, 1$. Define the (i, α) -*face* of σ as the smooth map $\sigma_{(i, \alpha)}$ given by

$$\sigma_{(i, \alpha)}(x^1, \dots, x^k) = \sigma(x^1, \dots, x^{i-1}, \alpha, x^i, \dots, x^k).$$

Moreover, define the *boundary* of σ as the $(k-1)$ -chain

$$\partial\sigma \equiv \sum_{i=1}^k (-1)^{i+1} (\sigma_{(i, 1)} - \sigma_{(i, 0)}).$$

3. If $\sigma := \sum_{i=1}^N a_i \sigma_i$ is a singular k -chain, then define the *boundary* of σ as the $(k-1)$ -chain

$$\partial\sigma \equiv \sum_{i=1}^N a_i \partial\sigma_i.$$

Note that $\int_{\partial\sigma} \omega = \sum_{i=1}^N a_i \int_{\partial\sigma_i} \omega$.

Definition 6.1.5. A singular k -chain σ is a *closed* if $\partial\sigma = 0$.

Exercise 6.1.6. Show that if σ is any singular k -chain, then $\partial\sigma$ is closed.

Theorem 6.1.7 (Stokes's theorem for chains). Let σ be a k -chain and $\omega \in \mathcal{A}^{k-1}(M)$. Then

$$\int_{\sigma} d\omega = \int_{\partial\sigma} \omega.$$

Proof. For now, assume that $M = \mathbb{R}^k$ and $\sigma = I^k$. As the smooth structure on \mathbb{R}^k is global, we may write $\omega = f dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$ for some distinguished $1 \leq i \leq k$ and some smooth function $f : \mathbb{R}^k \rightarrow \mathbb{R}$. We compute

$$\begin{aligned} d\omega &= df \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k \\ &= \left(\sum_{j=1}^k \frac{\partial f}{\partial x^j} dx^j \right) \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k \\ &= (-1)^{i-1} \frac{\partial f}{\partial x^i} dx^1 \wedge \dots \wedge dx^k. \end{aligned}$$

Now, apply Fubini's theorem together with the fundamental theorem of calculus (FTC) to obtain

$$\begin{aligned} \int_{\sigma} d\omega &= (-1)^{i-1} \int_{[0,1]^k} \frac{\partial f}{\partial x^i} dx^1 \wedge \dots \wedge dx^k \\ &= (-1)^{i-1} \int_0^1 \dots \int_0^1 \left(\int_0^1 \frac{\partial f}{\partial x^i} dx^i \right) dx^1 \dots \widehat{dx^i} \dots dx^k \\ &= (-1)^{i-1} \int_0^1 \dots \int_0^1 (f(x^1, \dots, \underbrace{1}_{i\text{-th position}}, \dots, x^k) - f(x^1, \dots, \underbrace{0}_{i\text{-th position}}, \dots, x^k)) dx^1 \dots \widehat{dx^i} \dots dx^k \\ &= (-1)^{i-1} \left(\int_{[0,1]^{k-1}} f(x^1, \dots, 1, \dots, x^k) dx^1 \dots \widehat{dx^i} \dots dx^k - \int_{[0,1]^{k-1}} f(x^1, \dots, 0, \dots, x^k) dx^1 \dots \widehat{dx^i} \dots dx^k \right) \\ &= (-1)^{i-1} \left(\int_{\sigma_{(i, 1)}} \omega - \int_{\sigma_{(i, 0)}} \omega \right). \end{aligned}$$

Moreover, we compute

$$\int_{\partial\sigma} \omega = \sum_{j=1}^k (-1)^{j-1} \left(\int_{\sigma_{(j,1)}} \omega - \int_{\sigma_{(j,0)}} \omega \right).$$

Since x^j is constant along the (j, α) -face for each $\alpha = 0, 1$, it follows that $dx^j = 0$. Therefore,

$$\int_{\partial\sigma} \omega = (-1)^{i-1} \left(\int_{\sigma_{(i,1)}} \omega - \int_{\sigma_{(i,0)}} \omega \right) = \int_{\sigma} d\omega.$$

Finally, assume that M is arbitrary and σ is an arbitrary k -cell on M . By the special case just proved, we have that

$$\int_{\sigma} d\omega = \int_{[0,1]^k} \sigma^*(d\omega) = \int_{[0,1]^k} d(\sigma^*\omega) = \int_{\partial[0,1]^k} \sigma^*\omega = \int_{\partial\sigma} \omega.$$

This clearly remains true if σ is a k -chain on M . \square

The FTC occurs precisely when $\sigma = I^1$ and $\omega = f$. This shows that Theorem 6.1.7 is equivalent to the FTC.

6.2 Lecture 23

Let M be an orientable manifold. Let $\omega \in \mathcal{A}^n(M)$. Let σ_1 and σ_2 be singular n -cells on M that can be extended to diffeomorphisms on (open) neighborhoods of $[0, 1]^n$. Suppose that both are orientation-preserving.

Lemma 6.2.1. *If $\text{supp } \omega \subset \sigma_1([0, 1]^n) \cap \sigma_2([0, 1]^n)$, then $\int_{\sigma_1} \omega = \int_{\sigma_2} \omega$.*

Proof. Since $\text{supp } \omega \subset \sigma_1([0, 1]^n) \cap \sigma_2([0, 1]^n)$, Proposition 6.1.3 implies that

$$\int_{\sigma_1} \omega = \int_{\sigma_2 \circ (\sigma_2^{-1} \circ \sigma_1)} \omega = \int_{\sigma_2} \omega.$$

\square

Let $\omega \in \mathcal{A}^n(M)$. Let σ be an orientation-preserving singular n -cell on M . If $\text{supp } \omega \subset \sigma([0, 1]^n)$, then Lemma 6.2.1 allows us to define

$$\int_M \omega = \int_{\sigma} \omega.$$

In general, there exists an open cover (U_α) of M such that $U_\alpha \subset \sigma_\alpha([0, 1]^n)$ for each α where σ_α is some orientation-preserving singular n -cell on M . Find a partition of unity (φ_α) subordinate to this cover. Note that each $\varphi_\alpha \omega$ belongs to $\mathcal{A}^n(M)$ and is supported in U_α . If ω is compactly supported, then $\text{supp } \omega$ intersects at most finitely many $\text{supp } \varphi_\alpha$. In this case, we define

$$\int_M \omega = \sum_{\alpha} \int_M \varphi_\alpha \omega,$$

which is finite. It remains to check that this definition makes sense.

Lemma 6.2.2. *If (V_β, ψ_β) is another such partition of unity, then $\sum_{\beta} \int_M \psi_\beta \omega = \sum_{\alpha} \int_M \varphi_\alpha \omega$.*

Proof.

$$\begin{aligned}
\sum_{\alpha} \int_M \varphi_{\alpha} \omega &= \sum_{\alpha} \int_M \varphi_{\alpha} \sum_{\beta} \psi_{\beta} \omega \\
&= \sum_{\alpha} \sum_{\beta} \int_M \varphi_{\alpha} \psi_{\beta} \omega \\
&= \sum_{\beta} \sum_{\alpha} \int_M \psi_{\beta} \varphi_{\alpha} \omega \\
&= \sum_{\beta} \int_M \psi_{\beta} \sum_{\alpha} \varphi_{\alpha} \omega \\
&= \sum_{\beta} \int_M \psi_{\beta} \omega.
\end{aligned}$$

□

Note 6.2.3. If ω is not assumed to be compact, then $\int_M \omega$ may be infinite but is still well-defined.

Theorem 6.2.4 (Stokes). *Let M be an oriented compact n -manifold with boundary. If $\omega \in \mathcal{A}^{n-1}(M)$, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. There are three cases to consider.

Case 1: Suppose that there is some orientation-preserving n -cell σ on M such that $\text{supp } \omega \subset \text{Int}(\text{im } \sigma)$ and $\text{im } \sigma \cap \partial M = \emptyset$. By Theorem 6.1.7, it follows that

$$\int_M d\omega = \int_{\sigma} d\omega = \int_{\partial\sigma} \omega = 0 = \int_{\partial M} \omega.$$

Case 2: Suppose that there is some orientation-preserving n -cell σ on M such that $\text{supp } \omega \subset \text{im } \sigma$, $\text{im } \sigma \cap \partial M = \sigma_{(n,0)}([0, 1]^{n-1})$, and $\text{supp } \omega \cap \text{im } \partial\sigma \subset \sigma_{(n,0)}$. By Theorem 6.1.7, it follows that

$$\int_M d\omega = \int_{\sigma} d\omega = \int_{\partial\sigma} \omega = (-1)^n \int_{\sigma_{(n,0)}} \omega.$$

Note that if μ denotes the usual orientation on \mathbb{H}^n , then the induced orientation on the boundary $\partial\mathbb{H}^n$ is equal to $(-1)^n \mu$. Therefore, $\sigma_{(n,0)} : [0, 1]^{n-1} \rightarrow \partial M$ is orientation-preserving if and only if n is even. In either situation, we have that

$$(-1)^n \int_{\sigma_{(n,0)}} \omega = \int_{\partial M} \omega,$$

which completes our present case.

Case 3: In general, there exist an open cover (U_{α}) of M and a partition of unity (φ_{α}) subordinate to it such that each $\varphi_{\alpha} \omega$ is an $(n-1)$ -form of the kind in Case 1 or Case 2. Since $\sum_{\alpha} \varphi_{\alpha}$ is constant, we see that

$$0 = d\left(\sum_{\alpha} \varphi_{\alpha}\right) = \sum_{\alpha} d\varphi_{\alpha}.$$

Hence $\sum_{\alpha} d\varphi_{\alpha} \wedge \omega = 0$, so that $\sum_{\alpha} \int_M d\varphi_{\alpha} \wedge \omega = 0$. From this we compute

$$\begin{aligned}
\int_M d\omega &= \int_M \sum_{\alpha} \varphi_{\alpha} d\omega \\
&= \sum_{\alpha} \int_M \varphi_{\alpha} d\omega \\
&= \sum_{\alpha} \int_M d\varphi_{\alpha} \wedge \omega + \varphi_{\alpha} d\omega \\
&= \sum_{\alpha} \int_M d(\varphi_{\alpha} \omega) \\
&= \sum_{\alpha} \int_{\partial M} \varphi_{\alpha} \omega \\
&= \int_{\partial M} \omega.
\end{aligned}$$

□

7 De Rham cohomology

7.1 Lecture 24

Given a smooth manifold M^n and integer $k \geq 1$, consider the vector spaces

$$\begin{aligned}
Z^k(M) &:= \{\omega \in \mathcal{A}^k(M) : d\omega = 0\} \\
B^k(M) &:= \{d\eta : \eta \in \mathcal{A}^{k-1}(M)\}.
\end{aligned}$$

Since $B^k(M) \subset Z^k(M)$, we may form the quotient space

$$H_{\text{dR}}^k(M) := Z^k(M) / B^k(M),$$

called the k -th de Rham cohomology group of M .

Remark 7.1.1. This is the same as the singular cohomology group over \mathbb{R} .

$H_{\text{dR}}^k(M)$ can be thought of as a quantitative measure of the number of submanifolds of M over which we can't integrate certain closed forms to find a potentials for them. In this sense, the failure of a closed form to be exact indicates holes in M .

Theorem 7.1.2. *If M and N are continuously homotopy equivalent, then $H_{\text{dR}}^k(M) \cong H_{\text{dR}}^k(N)$ for each $k \geq 1$.*

Recall that a space X is *contractible* if id_X is smoothly homotopic to the constant map at some point in X .

Lemma 7.1.3 (Poincaré). *If M is contractible, then $H_{\text{dR}}^k(M) = 0$ for each $k \geq 1$.*

Proof. For simplicity, assume that $k = 1$. For each $t \in [0, 1]$, define $\iota_t : M \rightarrow M \times [0, 1]$ by $p \mapsto (p, t)$.

Claim. *If ω is any closed 1-form on $M \times [0, 1]$, then $\iota_1^* \omega - \iota_0^* \omega$ is exact.*

Proof. If $\pi_M : M \times [0, 1] \rightarrow M$ denotes the projection and (U, x^i) denotes local coordinates on M , then $(\pi_M^{-1}(U), (\bar{x}^i, t))$ is a coordinate chart on $M \times [0, 1]$ where $\bar{x}^i := x^i \circ \pi_M$. We thus have that $\omega = w_i d\bar{x}^i + f dt$. For each $\alpha \in \{0, 1\}$, we see that

$$\iota_\alpha^* \omega = \iota_\alpha^*(w_i d\bar{x}^i + f dt) = w_i(-, \alpha) dx^i + 0.$$

Moreover,

$$\begin{aligned} 0 &= d\omega \\ &= dw_i \wedge d\bar{x}^i + df \wedge dt \\ &= (\text{terms not involving } dt) \\ &\quad + \frac{\partial w_i}{\partial t} dt \wedge d\bar{x}^i + \frac{\partial f}{\partial \bar{x}^i} d\bar{x}^i \wedge dt. \end{aligned}$$

This implies that $\frac{\partial w_i}{\partial t} = \frac{\partial f}{\partial \bar{x}^i}$ for each i . For each $p \in U$, we compute the sum

$$w_i(p, 1) - w_i(p, 0) = \int_0^1 \frac{\partial w_i}{\partial t}(p, t) dt = \int_0^1 \frac{\partial f}{\partial \bar{x}^i}(p, t) dt.$$

As a result,

$$\iota_1^* \omega - \iota_0^* \omega = \left(\int_0^1 \frac{\partial f}{\partial \bar{x}^i}(p, t) dt \right) dx^i.$$

Now, define $g : U \rightarrow \mathbb{R}$ by $\int_0^1 f(p, t) dt$, so that

$$\frac{\partial g}{\partial x^i} = \int_0^1 \frac{\partial f}{\partial \bar{x}^i}(p, t) dt.$$

It follows that $\iota_1^* \omega - \iota_0^* \omega = \frac{\partial g}{\partial x^i} dx^i = dg$. Since the pullback is coordinate-independent, g must be as well. This completes our proof. \square

By assumption, there is some smooth map $H : M \times [0, 1] \rightarrow M$ such that $H \circ \iota_1 = \text{id}_M$ and $H \circ \iota_0 = e_{p_0}$ where $p_0 \in M$. Let ω be a closed 1-form on M . Then $H^* \omega$ is closed since pullbacks commute with exterior derivatives. Recall that the pullback is a contravariant functor, giving us

$$\iota_k^* H^* \omega = (H \circ \iota_k)^* \omega$$

for each $k = 0, 1$. By our claim, it follows that

$$\iota_1^* H^* \omega - \iota_0^* H^* \omega = \omega - 0 = \omega$$

is closed. \square

The generalization of this result to any positive integer k proceeds as follows.

We have the decomposition

$$T_{(p,t)} M \times [0, 1] = \ker d\pi|_{(p,t)} \oplus \ker d\pi_M|_{(p,t)}$$

where $\pi : M \times [0, 1] \rightarrow [0, 1]$ denotes projection. Then any 1-form ω on $M \times [0, 1]$ may be written uniquely as $\omega = \omega_1 + \omega_2$ such that $\omega_i(v_1 + v_2) = \omega(v_i)$ for each $i = 1, 2$. Hence there is some unique map $f : M \times [0, 1] \rightarrow \mathbb{R}$

such that $\omega_2 = f dt$. In general, one can show that if ω is a k -form on $M \times [0, 1]$, then we can write ω uniquely as

$$\omega = \omega_1 + (dt \wedge \eta)$$

where $\omega_1(v_1, \dots, v_k) = 0$ if some v_i belongs to $\ker d\pi_M|_{(p,t)}$ and η is a $(k-1)$ -form with the analogous property.

Lemma 7.1.4. *Define the $(k-1)$ -form $I\omega$ on M by*

$$I\omega|_p(v_1, \dots, v_{k-1}) = \int_0^1 \eta(p, t) \left(dt_t|_{(p,t)}(v_1), \dots, dt_t|_{(p,t)}(v_{k-1}) \right) dt.$$

Then $\iota_1^ \omega - \iota_0^* \omega = d(I\omega) + I(d\omega)$. In particular, $\iota_1^* \omega - \iota_0^* \omega$ is exact whenever $d\omega = 0$.*

Proof. For an argument similar to our case where $k = 1$, see Theorem 7.17 (Spivak). In particular, $I\omega$ and η correspond to our g and f , respectively. \square

Corollary 7.1.5. *Recalling (†), we see that $\mathbb{R}^2 \setminus \{0\}$ is not contractible.*

This proves that $\mathbb{R}^2 \setminus \{0\} \not\approx \mathbb{R}^2$.

7.2 Lecture 25

Corollary 7.2.1. *If M is closed (i.e., compact without boundary) and orientable, then M is not contractible.*

Proof. There is some positively oriented orientation form ω on M . Then $d\omega = 0$, and $\int_M \omega > 0$. But if $\omega = d\eta$ for some form η , then $\int_M \omega = \int_{\partial M} \eta = 0$ thanks to Theorem 6.2.4, a contradiction. Hence $H^n(M) \neq 0$. \square

Example 7.2.2. \mathbb{S}^n is not contractible.

Theorem 7.2.3. *If M is a (connected) orientable n -manifold, then we have an isomorphism*

$$\underbrace{H_c^n(M)}_{\text{compactly supported}} \xrightarrow{\cong} \mathbb{R}, \quad [\omega] \mapsto \int_M \omega.$$

Proof. Assume that this statement holds when $M = \mathbb{R}^n$. There is some compactly supported orientation form ω on M such that $\int_M \omega \neq 0$ and $\text{supp } \omega \subset U \subset M$. Let ω' be a compactly supported n -form on M . Pick a partition of unity (φ_α) on M . Then $\omega' = \varphi_1 \omega' + \dots + \varphi_k \omega'$. Thus, we may assume that $\text{supp } \omega' \subset V$ where $V \approx \mathbb{R}^n$. We want to show that $\omega' = c\omega + d\eta$ for some $c \in \mathbb{R}$ and some $\eta \in \mathcal{A}^{n-1}(M)$. Since M is connected, there is some sequence

$$U = V_1, V_2, \dots, V_r = V$$

of open sets such that $V_i \approx \mathbb{R}^n$ and $V_i \cap V_{i+1} \neq \emptyset$ for each $i = 1, \dots, r-1$. We can find a family $\{\omega_i\}_{1 \leq i \leq r-1}$ of forms on M such $\int_M \omega_i \neq 0$ and $\text{supp } \omega_i \subset V_i \cap V_{i+1}$. It follows that

$$\begin{aligned} \omega_1 &= c_1 \omega + d\eta_1 \\ \omega_2 &= c_2 \omega_1 + d\eta_2 \\ &\vdots \\ \omega' &= c_r \omega_{r-1} + d\eta_r, \end{aligned}$$

as desired. \square

If M and N are closed orientable n -manifolds and $f : M \rightarrow N$ is smooth, then the pullback f^* induces a linear map $f^* : H_{\text{dR}}^n(N) \rightarrow H_{\text{dR}}^n(M)$. In light of Theorem 7.2.3, we get a linear map $f^* : \mathbb{R} \rightarrow \mathbb{R}$, which shows that there is a unique real number a such that

$$\int_M f^* \omega = a \int_N \omega$$

for every $\omega \in H_{\text{dR}}^n(N)$. The scalar a is called the *degree of f* .

7.3 Lecture 26

Let M and N be closed orientable n -manifolds and $f : M \rightarrow N$ be smooth. By Theorem 3.6.6, find some regular value q of f . For each $p \in f^{-1}(q)$, let

$$\text{sgn}_p f = \begin{cases} 1 & df_p \text{ orientation-preserving} \\ -1 & df_p \text{ orientation-reversing} \end{cases}.$$

Theorem 7.3.1.

$$\text{deg } f = \sum_{p \in f^{-1}(q)} \text{sgn}_p f$$

where $\text{deg } f \equiv 0$ if $f^{-1}(q) = \emptyset$. In particular, $\text{deg } f$ is always an integer.

Proof. Since f has constant rank n and $\{q\} \subset N$ is compact, we see that $f^{-1}(q)$ is a compact 0-dimensional submanifold of M by Theorem 3.6.4 and thus must be finite. Let $f^{-1}(q) = \{p_1, \dots, p_k\}$. Find charts U_1, \dots, U_k which are pairwise disjoint so that each $u_i \in U_i$ is a regular point of f . Find a chart (V, y^i) around q such that the components of $f^{-1}(V)$ are precisely the U_i . Let $\omega = g dy^1 \wedge \dots \wedge dy^n$ where g is nonnegative and compactly supported in V . This implies that $f^* \omega \subset f^{-1}(V) = U_1 \sqcup \dots \sqcup U_k$. Therefore,

$$\int_M f^* \omega = \sum_{i=1}^k \int_{U_i} f^* \omega.$$

Since each $f|_{U_i} : U_i \rightarrow V$ is a diffeomorphism, we have that

$$\int_{U_i} f^* \omega = \begin{cases} \int_V \omega & f|_{U_i} \text{ orientation-preserving} \\ -\int_V \omega & f|_{U_i} \text{ orientation-reversing} \end{cases}.$$

As a result,

$$\int_M f^* \omega = \left(\sum_{p \in f^{-1}(q)} \text{sgn}_p f \right) \int_V \omega = \left(\sum_{p \in f^{-1}(q)} \text{sgn}_p f \right) \int_M \omega.$$

□

Example 7.3.2. Let $A_n : \mathbb{S}^n \rightarrow \mathbb{S}^n$ denote the antipodal map. Choose $p_0 \in \mathbb{S}^n$, which is a regular value of A_n . Hence $\text{deg } A_n = (-1)^{n-1}$.

Theorem 7.3.3. Suppose that f and g are smoothly homotopic maps $M \rightarrow N$. Then $f^* = g^*$ as linear maps.

Proof. By assumption, there exists a smooth map $H : M \times [0, 1] \rightarrow M$ such that $H \circ \iota_0 = f$ and $H \circ \iota_1 = g$. Let $\omega \in Z^k(N)$. We apply Lemma 7.1.4 (including its notation) to compute

$$\begin{aligned} g^*\omega - f^*\omega &= (H \circ \iota_1)^*\omega - (H \circ \iota_0)^*\omega \\ &= \iota_1^*(H^*\omega) - \iota_0^*(H^*\omega) \\ &= d(IH^*\omega) + I(dH^*\omega) = d(IH^*\omega). \end{aligned}$$

This implies that $f^*([\omega]) = g^*([\omega])$, as desired. \square

Corollary 7.3.4. *If f and g are smoothly homotopic, then $\int_M f^*\omega = \int_M g^*\omega$ for any closed n -form ω .*

Proof. By Theorem 7.3.3, $f^*\omega = g^*\omega + d\eta$ for some $(n-1)$ -form η . Since M is closed by hypothesis, applying \int to both sides and then invoking Stokes's theorem finishes our proof. \square

Corollary 7.3.5. *If f and g are smoothly homotopic, then $\deg f = \deg g$.*

Corollary 7.3.6 (Hairy ball). *If $n \in \mathbb{N}$ is even, then there is no non-vanishing vector field on \mathbb{S}^n .*

Proof. The identity map $\text{id}_{\mathbb{S}^n}$ has degree 1 and thus is not homotopic to the antipodal map A_n . Suppose, toward a contradiction, that there is some non-vanishing $X \in \mathcal{X}(\mathbb{S}^n)$. For each $p \in \mathbb{S}^n$, there is a unique great semicircle γ_p traveling from p to $A(p)$ whose tangent vector at p equals cX_p for some $c \in \mathbb{R}$. The smooth map $H(p, t) \equiv \gamma_p(t)$ defines a homotopy between $\text{id}_{\mathbb{S}^n}$ and A_n , a contradiction. \square

8 Integral curves and flows

8.1 Lecture 27

Definition 8.1.1. Let M be a smooth manifold and $X \in \mathcal{X}(M)$. We say that a differentiable curve $\gamma : J \rightarrow M$ is an *integral curve for X* if $\gamma'(t) = X_{\gamma(t)}$ for any $t \in J$.

Terminology. If $0 \in J$, then $\gamma(0)$ is called the *starting point of γ* .

Example 8.1.2. Let $M = \mathbb{R}^2$, $X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$, and $\gamma(t) = (x(t), y(t))$. Then $\gamma'(t) = x'(t) \frac{\partial}{\partial x} + y'(t) \frac{\partial}{\partial y}$. The system

$$\begin{cases} x(t) = x'(t) \\ y(t) = y'(t) \end{cases}$$

determines that $\gamma(t) = e^t(x(0), y(0))$.

In general, define the vector field $x^i \frac{\partial}{\partial x^i}$ on a chart (U, x^i) for the n -manifold M . Then given an integral curve $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$ for X where $\gamma^i = \gamma \circ x^i$, we obtain the system

$$\gamma'^i(t) = X^i(\gamma^1(t), \dots, \gamma^n(t)).$$

Given that $\gamma(0) = p$, we have an initial value problem, to which we can always find a *local* solution.

Theorem 8.1.3 (Fundamental theorem for autonomous ODEs). *Let $U \subset \mathbb{R}^n$ be open and $X : U \rightarrow \mathbb{R}^n$ be a smooth vector field. Consider the initial value problem*

$$\begin{cases} \gamma^n(t) = X^i(\gamma^1(t), \dots, \gamma^n(t)) \\ \gamma(t_0) = (c^1, \dots, c^n) \end{cases} \quad (1)$$

(a) (Existence) *Let $t_0 \in \mathbb{R}$ and $x_0 \in U$. There exist some interval $J_0 \ni t_0$ and open subset $U_0 \subset U$ such that for each $c \in U_0$, there is some C^1 curve $\gamma : J_0 \rightarrow U_0$ that solves Eq. (1).*

(b) (Uniqueness) *Any two differentiable solutions to Eq. (1) agree on their common domain.*

(c) (Smoothness) *Let J_0 and U_0 be as in (a). Define $\theta : J_0 \times U_0 \rightarrow U$ by $(t, x) \mapsto \gamma_x(t)$ where $\gamma_x : J_0 \rightarrow U$ uniquely solves Eq. (1) with initial condition $\gamma(t_0) = x$. Then θ is smooth.*

Example 8.1.4. For any compact manifold M , we may stipulate that the U_0 form a finite cover $\{U_1, \dots, U_k\}$ of M . Make J_0 smaller than any of the corresponding intervals J_1, \dots, J_k . This yields a smooth map $\theta : J \times \mathbb{S}^n \rightarrow \mathbb{S}^n$ defined by $(t, p) \mapsto \gamma_p^i(t)$.

Corollary 8.1.5. *Let X be a smooth vector field on M and $p \in M$. There is some $\epsilon > 0$ along with a smooth curve $\gamma : (-\epsilon, \epsilon) \rightarrow M$ such that $\gamma(0) = p$ and γ is an integral curve for X .*

Definition 8.1.6. Let $\theta : \mathbb{R} \times M \rightarrow M$ be a group action on M .

1. We call θ a *global flow on M* if it is smooth, i.e., $\theta^p(t) := \theta(t, p) : \mathbb{R} \rightarrow M$ is smooth for every $p \in M$.
2. We call the vector field $p \mapsto (\theta^p)'(0)$ the *infinitesimal generator of θ* .

Question. When is a smooth vector field an infinitesimal generator of a global flow?

Example 8.1.7. Define $X = x^3 \frac{\partial}{\partial x}$ on \mathbb{R}^2 . Then any integral curve $\gamma(t) = (x(t), y(t))$ for X must satisfy

$$\begin{aligned} \frac{dx}{dt} &= x^3 \implies dx = x^3 dt \\ &\implies -\frac{1}{2x^2} = t + c \\ &\implies x(t) = \frac{1}{\sqrt{c - 2t}}, \end{aligned}$$

which is not smooth on \mathbb{R} . Hence X fails to generate a global flow.

Lemma 8.1.8 (Escape). *Let $X \in \mathcal{X}(M)$ and γ be an integral curve for X . If the domain of γ is not equal to \mathbb{R} , then $\text{im } \gamma$ is not contained in any compact set.*

Remark 8.1.9. If M is compact, then every smooth vector field on M generates a global flow.

Definition 8.1.10. A *flow domain* for M is an open subset $D \subset \mathbb{R} \times M$ such that for every $p \in M$, the set $\{t \in \mathbb{R} \mid (t, p) \in D\}$ is an open interval containing 0

Theorem 8.1.11 (Fundamental theorem on flows). *Let M be a smooth manifold and $X \in \mathcal{X}(M)$. There exist some unique maximal flow domain $\mathcal{D} \subset \mathbb{R} \times M$ and unique flow $\varphi : \mathcal{D} \rightarrow M$ such that X generates φ .*

Terminology. We call φ the *flow of X* .

Corollary 8.1.12. *If M is a closed manifold, then $\mathcal{D} = \mathbb{R} \times M$.*

8.2 Lecture 28

Let M be a smooth manifold without boundary. Let $V \in \mathcal{X}(M)$ and let θ denote the flow of V . For any $W \in \mathcal{X}(M)$, define the section of TM by

$$(\mathcal{L}_V W)_p \equiv \lim_{t \rightarrow 0} \frac{d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) - W_p}{t},$$

which always exists. This is called the *Lie derivative of W with respect to V* .

Proposition 8.2.1. $\mathcal{L}_V W \in \mathcal{X}(M)$.

We can view the Lie derivative at a point p as the rate of change of W along the tangent vector $V|_p$.

Theorem 8.2.2. If $V, W \in \mathcal{X}(M)$, then $\mathcal{L}_V W = [V, W]$.

Proof. Let $\mathcal{R}(M)$ denote the set of points $p \in M$ such that $V_p \neq 0$. Note that $\text{cl}(\mathcal{R}(M)) = \text{supp } V$. Let $p \in M$. We have three cases to consider.

- (i) Suppose that $p \in \mathcal{R}(M)$. We can find smooth coordinates (U, u^i) near p such that $V = \frac{\partial}{\partial u^1}$. In these coordinates we thus have that $\theta_t(u) = (u^1 + t, u^2, \dots, u^n)$. The Jacobian of θ_{-t} at each t equals the identity. For any $u \in U$, it follows that

$$\begin{aligned} & d(\theta_{-t})_{\theta_t(u)}(W_{\theta_t(u)}) \\ &= d(\theta_{-t})_{\theta_t(u)} \left(W^j(u^1 + t, u^2, \dots, u^n) \frac{\partial}{\partial u^j} \Big|_{\theta_t(u)} \right) \\ &= W^j(u^1 + t, u^2, \dots, u^n) \frac{\partial}{\partial u^j} \Big|_u. \end{aligned}$$

From this we compute

$$\begin{aligned} (\mathcal{L}_V W)_p &= \frac{d}{dt} \Big|_{t=0} W^j(u^1 + t, u^2, \dots, u^n) \frac{\partial}{\partial u^j} \Big|_u \\ &= \frac{\partial}{\partial u^1} W^j(u^1, u^2, \dots, u^n) \frac{\partial}{\partial u^j} \Big|_u \\ &= [V, W]_u. \end{aligned}$$

- (ii) Suppose that $p \in \text{supp } V \setminus \mathcal{R}(M)$. Since $\text{supp } V$ is dense in M and TM is Hausdorff, it follows that $(\mathcal{L}_V W)_p = [V, W]_p$.

- (iii) If $p \in M \setminus \text{supp } V$, then V vanishes on some neighborhood H of p . This implies that $\theta_t = \text{id}_H$, so that $d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = W_p$. Hence $(\mathcal{L}_V W)_p = 0 = [V, W]_p$.

□

Definition 8.2.3. A smooth local frame (X_1, \dots, X_n) is called a *commuting* or *holonomic frame* if $[X_i, X_j] = 0$ for any $1 \leq i, j \leq n$.

Theorem 8.2.4. Let M be a smooth n -manifold. Let (X_1, \dots, X_k) be a linearly independent k -tuple of smooth commuting vector fields defined on an open set $W \subset M$. For any $p \in W$, there is some chart (U, x^i) around p such that

$$X_i = \frac{\partial}{\partial x^i}$$

on U for each $i = 1, \dots, k$.

Proof sketch. As this statement is local, we may assume that $M = \mathbb{R}^n$ and $p = 0$. Since the X_i are linearly independent, we can find coordinates (V, t^i) around 0 such that $X_i|_0 = \frac{\partial}{\partial t^i}|_0$ for each i . Let θ^i denote the flow of X_i . By making V a sufficiently small neighborhood of 0 in $\mathbb{R}^k \times \mathbb{R}^{n-k} \approx \mathbb{R}^n$, define $\Psi : V \rightarrow \mathbb{R}^n$ by

$$\Psi(t^1, \dots, t^n) = \theta_{t^1}^1 \circ \dots \circ \theta_{t^k}^k (0, \dots, 0, t^{k+1}, \dots, t^n).$$

Since the X_i are commuting, one can show that

$$d\Psi_0 = \begin{cases} X_i|_0 & i = 1, \dots, k \\ \frac{\partial}{\partial t^i}|_0 & i = k+1, \dots, n. \end{cases}$$

This is invertible, and thus Ψ is a local diffeomorphism by the inverse function theorem. This gives us our desired local coordinates. \square

9 Distributions

Definition 9.0.1. Let M be a smooth manifold. A k -distribution on M is a rank- k smooth subbundle of TM .

In particular, 1-distributions are precisely vector fields.

Definition 9.0.2. Let $N \subset M$ be a nonempty submanifold and

$$D := \coprod_{p \in M} D_p$$

be a distribution on M . Then N is called an *integral manifold of D* if $D_p = T_p N$ for each $p \in N$. Moreover, we say that D is *integrable* if each $p \in M$ is contained in an integrable manifold of D .

Definition 9.0.3. We say that a distribution D is *involutive* if $[X, Y] \in D$ whenever $X, Y \in D$.

Proposition 9.0.4. *If D is integrable, then it is involutive.*

Theorem 9.0.5 (Frobenius). *If D is involutive, then it is integrable.*