Type-theoretic Brown representability

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Overview

- A *sketch* of a type-theoretic, eventually computer-checked proof of Brown representability

- Application of this to **classifying (reduced) cohomology theories inside HoTT**

Along the way, we'll address some subtleties particular to HoTT.

Classical Brown representability

Let **CW** denote the ∞-category of *pointed, connected* CW complexes.

Theorem

For every functor $F : \mathrm{Ho}(\mathbf{CW})^{\mathrm{op}} \to \mathbf{Set}$, *if* F *sends*

- *countable coproducts to products*
- *pushouts in* **CW** *to weak pullbacks,*

then F *is representable, i.e.,*

$$F(-) \cong \|- \to_* X\|_0$$

for some $X \in \mathbf{CW}$.

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Example

All *additive* cohomology theories are represented by Ω-spectra.

Brief history

- Brown (1962)

  the original theorem, for **CW**

- Brown (1965)

  a general version expressed in category theory

- **Lurie (2017)**

  another general version, expressed in a setting suitable for HoTT

Lurie's version

Let $\mathscr{C}$ be a locally presentable ∞-category.

Suppose that $\mathscr{C}$ is generated under colimits by a set $\{X_i\}_{i \in I}$ of objects of $\mathscr{C}$ where

- each X_i is *compact*;
- each X_i is a *cogroup*; and
- X is closed under suspensions.

Theorem (BRT)

For every functor $F : \mathrm{Ho}(\mathscr{C})^{\mathrm{op}} \to \mathbf{Set}$, if F sends

- *countable coproducts to products*
- *pushouts in $\mathscr{C}$ to weak pullbacks,*

then F is representable, i.e.,

$$F(-) \cong \| - \to_* A \|_0$$

for some $A \in \mathscr{C}$.

Translation to type theory:

1. Take any family $X : I \to \mathcal{U}^*$ of types such that
   - each X_i is *compact* in $\mathcal{U}^*$;
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2. Form the subtype $\mathrm{C}(X)$ of $\mathcal{U}^*$ consisting of all iterated *pointed* colimits of diagrams valued in X.

   Implemented as the total space of connected components of a particular inductive-recursive family $D \to \mathcal{U}$.

Let F be a (1-coherent) contravariant functor from $\mathcal{U}^*$ to **Set**.

Suppose that F sends

- countable wedge sums in $\mathsf{C}(X)$ to products of sets
- pushouts in $\mathsf{C}(X)$ to weak pullbacks of sets.

Then $F(-) \cong \|- \to_* A\|_0$ for some type A in $\mathsf{C}(X)$.

Construct an object A_X in $\mathsf{C}(X)$ that represents F on X.

$$R_0 \quad :\equiv \quad \bigvee_{i:I,x:F(X_i)} X_i$$

$$\begin{array}{ccc}
\bigvee_{i:I,x:\ker(\|X_i \to_* R_n\|_0 \to F(X_i))} X_i & \longrightarrow & R_n \\
\downarrow & & \downarrow \\
1 & \longrightarrow & R_{n+1}
\end{array}$$

$$A_X \quad :\equiv \quad \operatorname{colim}_{n:\mathbb{N}} R_n$$

To see that A_X represents F on *all* $\mathsf{C}(X)$, we need the following Yoneda-like lemma.

Lemma

For every A, B in $\mathsf{C}(X)$ and $(f, f_p) : A \to_* B$, if the function

$$\|X_i \to_* A\|_0 \xrightarrow{\|(f,f_p)\circ-\|_0} \|X_i \to_* B\|_0$$

is an equivalence for all $i : I$, then $A \xrightarrow{f} B$ is an equivalence.

This reduces to Whitehead's theorem for $\mathsf{C}(X)$ when $X \equiv \{S^n\}_{n\geq 1}$.

In this sense, we need $\mathsf{C}(X)$ to be *strongly generated* by X.

This may not be provable in general but is consistent to postulate.

Alternatively, we can modify $\mathsf{C}(X)$ so that it's strongly generated by X without extra axioms.

For example, for each truncation level k, instead take the subuniverse consisting of the k-truncations of all elements of $\mathsf{C}(X)$.

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Question: Can we prove that $\mathrm{C}(\{S^n\}_{n\geq 1})$ is strongly generated?

If not, how about the subuniverse of countable CW complexes?

Classifying cohomology theories

Two consequences of the BRT, due to Eilenberg and Steenrod:

1. *Every additive cohomology theory* $(\mathcal{U}^*)^{\mathrm{op}} \to \mathbf{Ab}$ *is represented on* $\mathsf{C}(X)$ *by a pre-spectrum valued in* $\mathsf{C}(X)$*.*

2. Take $X \equiv \{S^n\}_{n \geq 1}$. Let $h^\bullet, k^\bullet : (\mathcal{U}^*)^{\mathrm{op}} \to \mathbf{Ab}$ be additive cohomology theories.

   For each natural transformation $T^\bullet : h^\bullet \Rightarrow k^\bullet$*, if the map* $T^n(\mathbf{2}) : h^n(\mathbf{2}) \to k^n(\mathbf{2})$ *of abelian groups is an isomorphism for each* $n : \mathbb{N}$*, then* $T^\bullet \restriction_{\mathsf{C}(X)}$ *is an isomorphism.*

   If both $h^\bullet$ *and* $k^\bullet$ *are ordinary, then every isomorphism* $\tau : h^0(\mathbf{2}) \xrightarrow{\cong} k^0(\mathbf{2})$ *of abelian groups extends to an isomorphism* $h^\bullet \restriction_{\mathsf{C}(X)} \xrightarrow{\cong} k^\bullet \restriction_{\mathsf{C}(X)}$ *of cohomology theories.*

Inside HoTT, cohomology theories induced by spectra P need not be (countably) additive.

$$\left\| \bigvee_{k:\mathbb{N}} A_k \to_* P_n \right\|_0 \quad \text{vs.} \quad \prod_{k:\mathbb{N}} \| A_k \to_* P_n \|_0$$

Are there nontrivial additive cohomology theories inside HoTT?

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Are all homology theories induced by pre-spectra additive (at least, in enough cases) inside HoTT?

Can we construct non-trivial injective abelian groups inside HoTT?

References

Edgar Brown, *Cohomology Theories*, 1962

Edgar Brown, *Abstract Homotopy Theory*, 1965

Jacob Lurie, *Higher Algebra*, 2017