Coslice colimits in HoTT

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Overview

1. Background
2. Colimits in coslices of a universe
3. Interaction with factorization systems
4. Interaction with cohomology

Motivation

- Coslices of the category of spaces, including the category of pointed spaces, appear often in classical homotopy theory.

  For example, the Brown representability theorem is about homotopy functors on pointed connected spaces.

- These categories have natural roles in HoTT too, as do their colimits.

  We'd like to study the theory of *coslice colimits* (over free categories) inside HoTT, i.e., colimits in the wild category $A/\mathcal{U}$.

Goals

1. Construct coslice colimits so as to reveal their relation to *ordinary* colimits (i.e., colimits in $\mathcal{U}$).

2. Use such a contruction to extract useful information about coslice colimits.

3. Apply some of this information to factorization systems, higher group theory, and cohomology.

Formal setting

The main construction of coslice colimits is done in MLTT + Pushout.

Pushouts have definitional computation rules just on point constructors.

Some applications also use univalence.

A *graph* is a pair $\Gamma :\equiv (\Gamma_0, \Gamma_1)$ consisting of a type $\Gamma_0 : \mathcal{U}$ of vertices and a family $\Gamma_1 : \Gamma_0 \to \Gamma_0 \to \mathcal{U}$ of edges.

Let F be a Γ-shaped diagram in $\mathcal{U}$. The *(ordinary) colimit of F* is the HIT $\mathrm{colim}_\Gamma(F)$ generated by

$$\begin{aligned} \iota &: (i : \Gamma_0) \to F_i \to \mathrm{colim}_\Gamma(F) \\ \kappa &: (i, j : \Gamma_0)\,(g : \Gamma_1(i,j)) \to \iota_j \circ F_{i,j,g} \sim \iota_i \end{aligned}$$

$$\begin{tikzcd} F_i \arrow[rr, "F_{i,j,g}"] \arrow[dr, "\iota_i"'] \arrow[rr, phantom, shift right=5ex, "\kappa_{i,j,g}"] & & F_j \arrow[dl, "\iota_j"] \\ & \mathrm{colim}_\Gamma(F) & \end{tikzcd}$$

Coslice colimits

Let $A : \mathcal{U}$. We have a wild category $A/\mathcal{U}$ defined by

$$\begin{aligned}
\mathrm{Ob}(A/\mathcal{U}) &:= \sum_{X:\mathcal{U}} A \to X \\
\hom_{A/\mathcal{U}}(Y_1, Y_2) &:= \sum_{f:\mathrm{pr}_1(Y_1)\to\mathrm{pr}_1(Y_2)} f \circ \mathrm{pr}_2(Y_1) \sim \mathrm{pr}_2(Y_2)
\end{aligned}$$

Coslice colimits

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Let Γ be a graph and F be a Γ-shaped diagram in $A/\mathcal{U}$.

An F-cocone (h, H) on an object Y of $A/\mathcal{U}$ is *colimiting* if for each $X : \mathrm{Ob}(A/\mathcal{U})$, the evident function

$$\mathrm{postcomp}(h, H) \;:\; (Y \to_A X) \;\to\; \mathrm{Cocone}_F(X)$$

is an equivalence.

An A-homotopy $h_j \circ F_{i,j,g} \sim_A h_i$ looks like a homotopy $\eta_{i,j,g} : \mathrm{pr}_1(h_j) \circ \mathrm{pr}_1(F_{i,j,g}) \sim \mathrm{pr}_1(h_i)$ equipped with a path

$$\eta_{i,j,g}(\mathrm{pr}_2(F_i)(a)) \cdot \mathrm{pr}_2(h_i)(a) \;=\; \mathrm{ap}_{\mathrm{pr}_1(h_j)}(\mathrm{pr}_2(F_{i,j,g})(a)) \cdot \mathrm{pr}_2(h_j)(a)$$

for each $a : A$.

This family of 2-cells distinguishes the colimit of F, in $A/\mathcal{U}$, from $\mathrm{colim}_\Gamma(\mathcal{F}(F))$ ($\mathcal{F} :=$ forgetful functor).

Example

If $i \equiv j$ and $F_{i,j,g} \equiv \mathrm{id}_{F_i}$, then we get a filler

$$\eta(\mathrm{pr}_2(F_i)(a)) = \mathrm{refl}_{\mathrm{pr}_1(h_i)(\mathrm{pr}_2(F_i)(a))}$$

Question: How should we construct the colimit of F in $A/\mathcal{U}$?

1. We could directly define it as a 2-HIT.

   Sure, but unhelpful.

2. We could apply the forgetful functor $\mathcal{F}$ to F and then form the colimit in $\mathcal{U}$ of $\mathcal{F}(F)$ augemtned by $\mathrm{pr}_1(A) \to \bullet$.

   Won't work.

3. We could form the colimit $\mathrm{colim}_\Gamma(\mathcal{F}(F))$ and then attach 2-dimensional fillers to it.

   Works nicely.

Define $\operatorname{colim}_\Gamma A \xrightarrow{\psi} \operatorname{colim}_\Gamma(\mathcal{F}(F))$ by colimit induction, as the function induced by the cocone

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\iota_i \circ \mathrm{pr}_2(F_i)"'] & & A \arrow[dl, "\iota_j \circ \mathrm{pr}_2(F_j)"] \\
& \operatorname{colim}_\Gamma(\mathcal{F}(F)) &
\end{tikzcd}$$

under the constant diagram at A.

Intutively, this map finds those loops in $\operatorname{colim}_\Gamma(\mathcal{F}(F))$ that must be filled.

Form the pushout square

$$\begin{tikzcd}
\mathrm{colim}_\Gamma A \arrow[r, "\psi"] \arrow[d, "\langle \mathrm{id}_A \rangle_{i:\Gamma_0}"'] & \mathrm{colim}_\Gamma(\mathcal{F}(F)) \arrow[d, "\mathrm{inr}"] \\
A \arrow[r, "\mathrm{inl}"'] & \mathcal{P}_F \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

We can form an F-cocone $\mathcal{K}(\mathcal{P}_F)$ on $(\mathcal{P}_F, \mathrm{inl})$

$$\begin{tikzcd}
F_i \arrow[rr, "F_{i,j,g}"] \arrow[dr, "(\mathrm{inr} \circ \iota_i, \tau_i)"'] & \langle \delta_{i,j,g}, \epsilon_{i,j,g} \rangle & F_j \arrow[dl, "(\mathrm{inr} \circ \iota_j, \tau_j)"] \\
& \mathcal{P}_F &
\end{tikzcd} \qquad (\tau_i(a) :\equiv \mathrm{glue}_{\mathcal{P}_F}(\iota_i(a))^{-1})$$

as follows.

We have a homotopy

$$\delta_{i,j,g} :\equiv \lambda(x : \mathsf{pr}_1(F_i)).\, \mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x))$$

from $\mathsf{inr} \circ \iota_j \circ \mathsf{pr}_1(F_{i,j,g})$ to $\mathsf{inr} \circ \iota_i$.

For each $a : A$, we have a chain $\epsilon_{i,j,g}(a)$ of identities

$$\begin{aligned}
& \mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \tau_j(a) \\
= \; & \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{refl}_{\mathsf{inl}(a)} \\
= \; & \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{refl}_{\mathsf{inl}(a)} && (\textit{via } \rho_\psi(i,j,g,a)) \\
= \; & \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{ap}_{\langle \mathsf{id}_A \rangle}(\kappa_{i,j,g}(a))) \\
& && (\textit{via } \rho_{\langle \mathsf{id}_A \rangle}(i,j,g,a)) \\
= \; & (\kappa_{i,j,g}(a))_* (\tau_j(a)) && (\textit{transport on identity type}) \\
= \; & \tau_i(a) && (\textit{by } \mathsf{apd}_{\mathsf{glue}(-)^{-1}}(\kappa_{i,j,g}(a)))
\end{aligned}$$

Theorem (*(HoTT-)Agda formalized*)

The function

$$\mathrm{postcomp}(\mathcal{K}(\mathcal{P}_F), T) \;:\; ((\mathcal{P}_F, \mathrm{inl}) \to_A T) \;\to\; \mathrm{Cocone}_F(T)$$

is an equivalence for every $T : \mathrm{Ob}(A/\mathcal{U})$.

Corollary

The forgetful functor $A/\mathcal{U} \to \mathcal{U}$ *creates colimits over trees, i.e., graphs whose quotients are contractible.*

So far, we have defined a function

$$\mathrm{colim}_\Gamma^A := \mathcal{P} : \mathrm{Ob}(\mathrm{Diag}(\Gamma, A/\mathcal{U})) \to \mathrm{Ob}(A/\mathcal{U})$$

We now make $\mathcal{P}$ a functor by describing its action on maps of diagrams.

Goal:

Describe this action in terms of the action of the ordinary colimit functor by using the special form of $\mathcal{P}$'s object function.

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Describe this action in terms of the action of the ordinary colimit functor by using the special form of $\mathcal{P}$'s object function.

Suppose that F and G are Γ-shaped diagrams in $A/\mathcal{U}$. Consider a morphism $\delta : F \Rightarrow_A G$.

The triangle

$$\begin{tikzcd}
 & \operatorname{colim}_\Gamma A \arrow[dl, "\psi_F"'] \arrow[dr, "\psi_G"] & \\
\operatorname{colim}_\Gamma(\mathcal{F}(F)) \arrow[rr, "\bar\delta"'] & & \operatorname{colim}_\Gamma(\mathcal{F}(G))
\end{tikzcd}$$

commutes by induction on $\operatorname{colim}_\Gamma A$.

We thus have a map

$$\begin{tikzcd}
A \arrow[d, "\mathrm{id}"'] & \operatorname{colim}_\Gamma A \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}"] & \operatorname{colim}_\Gamma(\mathcal{F}(F)) \arrow[d, "\bar\delta"] \\
A & \operatorname{colim}_\Gamma A \arrow[l] \arrow[r] & \operatorname{colim}_\Gamma(\mathcal{F}(G))
\end{tikzcd}$$

of spans, which induces a map $\operatorname{colim}_\Gamma^A(\delta) : \mathcal{P}_F \to_A \mathcal{P}_G$ by the universal property of pushouts.

It remains to prove that $\mathrm{colim}^A_\Gamma \dashv \mathrm{const}_\Gamma$.

Lemma (*Agda formalized*)

For every map $h^* : T \to_A U$, the following square commutes:

$$\begin{tikzcd}
\mathrm{colim}^A_\Gamma(F) \to_A T \arrow[r, "h^* \circ -"] \arrow[d, "\mathrm{postcomp}_{F,T}"'] & \mathrm{colim}^A_\Gamma(F) \to_A U \arrow[d, "\mathrm{postcomp}_{F,U}"] \\
\mathrm{Cocone}_F(T) \arrow[r, "\mathrm{Cocone}_F(h^* \circ -)"'] & \mathrm{Cocone}_F(U)
\end{tikzcd}$$

Lemma (*Agda formalized*)

For every $T : \mathrm{Ob}(A/\mathcal{U})$ *and* $\delta : F \Rightarrow_A G$*, the following square commutes:*

$$\begin{tikzcd}
\mathrm{colim}^A_\Gamma(G) \to_A T \arrow[r, "-\circ \mathrm{colim}^A_\Gamma(\delta)"] \arrow[d, "\mathrm{postcomp}_{G,T}"'] & \mathrm{colim}^A_\Gamma(F) \to_A T \arrow[d, "\mathrm{postcomp}_{F,T}"] \\
\mathrm{Cocone}_G(T) \arrow[r, "\mathrm{Cocone}(T)(-\circ\delta)"'] & \mathrm{Cocone}_F(T)
\end{tikzcd}$$

The proof is easier for $\mathrm{postcomp}^{-1}_{F,\mathrm{colim}^A_\Gamma(G)}(K(\delta))$ than for $\mathrm{colim}^A_\Gamma(\delta)$, where $K(\delta)$ is the canonical cocone on $\mathcal{P}_G$ under F induced by δ.

Therefore, we decide to reduce the goal to an A-homotopy between the two maps.

Lifting pullback stability

- Colimits in $A/\mathcal{U}$ need not be pullback-stable.
- Colimits in $\mathcal{U}$ are pullback-stable.

  We can prove this directly or, I think, first prove that $\mathcal{U}$ is LCC.
- Thus, tree-shaped colimits in $A/\mathcal{U}$ are also pullback-stable.

Interaction with factorization systems on $\mathcal{U}$

Let $(\mathcal{L}, \mathcal{R})$ be an OFS on $\mathcal{U}$.

For all diagrams $F, G : \mathcal{D}_\Gamma :\equiv \mathrm{Diag}(\Gamma, \mathcal{U})$ and natural transformations $(H, \gamma) : F \Rightarrow G$, define the predicates

$$\begin{aligned} \widehat{\mathcal{L}}(H,\gamma) &:\equiv (i : \Gamma_0) \to \mathcal{L}(H_i) \\ \widehat{\mathcal{R}}(H,\gamma) &:\equiv (i : \Gamma_0) \to \mathcal{R}(H_i). \end{aligned}$$

Lemma

Let $Q : F \Rightarrow G$. The following type is contractible:

$$\mathrm{fact}_{\widehat{\mathcal{L}},\widehat{\mathcal{R}}}(Q) :\equiv \sum_{M:\mathcal{D}_\Gamma} \sum_{S:F\Rightarrow M} \sum_{T:M\Rightarrow G} (T \circ S = Q) \times \widehat{\mathcal{L}}(S) \times \widehat{\mathcal{R}}(T).$$

Corollary (*anonymous reviewer at HoTT/UF*)

The ordinary colimit functor colim_Γ *takes* $\widehat{\mathcal{L}}$ *to* $\mathcal{L}$.

For each $X, Y : \mathrm{Ob}(A/\mathcal{U})$, consider the predicate $\mathcal{L}_A(f, p) :\equiv \mathcal{L}(f)$ on $X \to_A Y$.

The functor colim_Γ^A takes $\widehat{\mathcal{L}}_A$ to $\mathcal{L}_A$.

Indeed, consider a map $\delta : \mathcal{A} \Rightarrow_A \mathcal{B}$ of A-diagrams. The underlying function of $\mathrm{colim}_\Gamma^A(\delta)$ is induced by the morphism

$$\begin{tikzcd} A \arrow[d, "\mathrm{id}"'] & \mathrm{colim}_\Gamma A \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{colim}_\Gamma(\mathcal{F}(\mathcal{A})) \arrow[d, "\bar{\delta}"] \\ A & \mathrm{colim}_\Gamma A \arrow[l] \arrow[r] & \mathrm{colim}_\Gamma(\mathcal{F}(\mathcal{B})) \end{tikzcd}$$

of spans. Thus, if δ is in $\widehat{\mathcal{L}}_A$, then all three vertical functions are in $\mathcal{L}$. Since a map of spans is a map of diagrams, we see that $\mathrm{colim}_\Gamma^A(\delta)$ is in $\mathcal{L}_A$.

Notice that $\mathrm{colim}^{\mathbf{1}}_{\Gamma}(\mathbf{1})$ is contractible.

Thus, if F is a diagram of pointed types over Γ such that each $\mathrm{pr}_1(F_i)$ is $(\mathcal{L}, \mathcal{R})$-connected, then the type $\mathrm{colim}^{\mathbf{1}}_{\Gamma}(F)$ is also $(\mathcal{L}, \mathcal{R})$-connected.

Examples

- Pointed colimits preserve modal types associated with the closed subtopos.
- The categories of higher groups (BDR) have colimits over graphs.
- The categories of higher pointed abelian groups (coslices of higher groups under the free higher groups on one generator) have colimits over graphs.

Interaction with cohomology

The 3×3 lemma (Licata and Brunerie) lets us transform our pushout contruction of coslice colimits to a new pushout contruction:

$$\begin{tikzcd}
\left(\bigvee_{i,j,g} \mathrm{pr}_1(F_i)\right) \vee \left(\bigvee_{i,j,g} \mathrm{pr}_1(F_i)\right) \arrow[r] \arrow[d, "\mathrm{id} \vee \mathrm{id}"'] & \bigvee_i \mathrm{pr}_1(F_i) \arrow[d] \\
\bigvee_{i,j,g} \mathrm{pr}_1(F_i) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{colim}^A_\Gamma(F)
\end{tikzcd}$$

Let F be a diagram of pointed types over a graph Γ. Let H^* be a cohomology theory.

The universal property of limits in **Ab** gives us a commuting triangle

$$\begin{tikzcd}
H^n(\mathrm{colim}^*_\Gamma(F)) \arrow[rr, dashed, "\Delta_F"] \arrow[dr, "H^n(\iota_i)"'] & & \lim_\Gamma H^n(F) \arrow[dl, "\mathrm{pr}_i"] \\
& H^n(F_i) &
\end{tikzcd}$$

for each $i : \Gamma_0$.

If Γ is finite, then we can combine our new contruction of colim^A_Γ with the Mayer-Vietoris sequence (Cavallo) to deduce that Δ_F is an epi in **Set**.

WIth the axiom of choice, we have that $H^n(\mathrm{colim}^*_\Gamma(F))$ is merely a weak limit of sets.

References

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