A mechanized characterization of coherent 2-groups

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MURI Meeting 2025

Plan

1. Generalize the categorical equivalence $\mathbf{Grp} \simeq \mathbf{Ptd}^{conn}_{\leq 1}$ to the case of 2-groups.

   Result: *"explicit" biequivalence between coherent 2-groups and pointed connected 2-types.*

2. Use univalence to get an identity between the $(2,1)$-category of coherent 2-groups and that of pointed connected 2-types.

   Consider some limitations of the induced identity.

3. Consider some questions.

Note: The mathematical development is entirely mechanized in a library based on HoTT-Agda.

The 1-dimensional case

1. The delooping of a set-level group G (Licata and Finster):

   the 1-truncated HIT $K(G,1)$ generated by

   - a point base : $K(G,1)$
   - a homomorphism loop : $G \to \Omega(K(G,1), \text{base})$.

   Theorem: loop is an isomorphism.

   By the encode-decode method.

2. Equivalence of categories (Buchholtz, van Doorn, and Rijke):

$$\mathbf{Grp} \underset{\Omega}{\overset{K(-,1)}{\rightleftarrows}} \mathbf{Ptd}_{\leq 1}^{conn}$$

2-groups

A *(coherent) 2-group* (Baez and Lauda) is a 1-type G with

- a neutral element e
- a binary operation $\otimes : G \to G \to G$, called the *tensor product*
- a right unitor ρ, a left unitor λ, and an associator α for $\otimes$ *(all in the sense of paths)*
- a triangle identity and a pentagon identity
- an *inverse* operation $(-)^{-1} : G \to G$
- paths $\mathsf{linv}_x : x^{-1} \otimes x = \mathsf{id}$ and $\mathsf{rinv}_x : x \otimes x^{-1} = \mathsf{id}$ for each $x : G$ such that linv and rinv satisfy two zig-zag identities.

Concise version: a monoidal univalent groupoid where every element has an adjoint equivalence.

Example

For every pointed 2-type X, the loop space $\Omega(X)$ equipped with path composition has the structure of a 2-group.

A 2-*group morphism* $G_1 \to G_2$ is a function $f_0 : G_1 \to G_2$ equipped with a family of paths $\mu_{x,y} : f_0(x) \otimes f_0(y) = f_0(x \otimes y)$ that respects the associator.

Justification for our short definition of 2-group morphism:

For each function $f_0 : G_1 \to G_2$ between the underlying types of 2-groups, the forgetful function

$$\textit{fully explicit notion on } f_0 \quad \to \quad \textit{short notion on } f_0$$

is an equivalence.

Delooping a 2-group

Let G be a 2-group.

Construct its delooping by generalizing the delooping $K(-, 1)$ of a set-level group.

Form the 2-truncated HIT $K_2(G)$ generated by

- a point base : $K_2(G)$
- a morphism of 2-groups loop : $G \to \Omega(K_2(G), \text{base})$

Note: $K_2(G)$ is a pointed connected 2-type.

Key feature: no path constructors for units or inverses.

Define codes : $K_2(G) \to \mathcal{U}_{\leq 1}$ by recursion on $K_2(G)$.

1. $\mathrm{codes}(\mathrm{base}) :\equiv G$
2. a 2-group morphism $\zeta : G \to \Omega(\mathcal{U}_{\leq 1}, G)$ defined as follows:
   - Define $\zeta_{\mathrm{map}} : G \to (G = G)$ by mapping g to the equivalence

$$\begin{aligned} \text{post-mult}_g &: G \xrightarrow{\simeq} G \\ \text{post-mult}_g(x) &:\equiv x \otimes g \end{aligned}$$

     and then applying univ to post-mult_g.
   - Both post-mult and univ are 2-group morphisms.

     We give ζ_{map} the composite structure.

Let $\mathrm{codes}_0 :\equiv \mathrm{pr}_1 \circ \mathrm{codes}$ and define

$$\begin{aligned} \mathrm{encode} &: \prod_{z:K_2(G)} \mathrm{base} = z \to \mathrm{codes}_0(z) \\ \mathrm{encode}(z, p) &:\equiv \mathrm{transp}^{\mathrm{codes}_0}(p, \mathrm{e}_G) \end{aligned}$$

Goal: Prove that

$$\mathtt{eb} :\equiv \mathrm{encode}(\mathrm{base}) \;:\; \Omega(K_2(G)) \to G$$

is inverse to loop.

The harder homotopy: $\mathrm{loop} \circ \mathtt{eb} \sim \mathrm{id}_{\mathrm{base}=\mathrm{base}}$.

Same strategy as 1-dimensional case:

Define

$$\mathrm{decode} \;:\; \prod_{z:K_2(G)} \mathrm{codes}_0(z) \to \mathrm{base} = z$$

by induction on $K_2(G)$ with $\mathrm{decode}(\mathrm{base}) :\equiv \mathrm{loop}$.

Then, by path induction, $\mathrm{decode}_z(\mathrm{encode}_z(p)) = p$ for all $z : K_2(G)$ and $p : \mathrm{base} = z$ as loop preserves the identity element.

The loop case is again an identity $\psi_{\mathsf{loop}}(x, y)$ of type

$$\mathsf{transp}^{\lambda z.\mathsf{base}=z}(\mathsf{loop}(x), \mathsf{loop}(y)) = \mathsf{loop}(\mathsf{transp}^{\mathsf{codes}_0}(\mathsf{loop}(x), y))$$

for all $x, y : G$.

The major difference from the 1-dimensional case:

The target of the $K_2(G)$-induction is a 1-type, not a set.

We need to construct a nontrivial coherence between various instances of ψ_{loop} and loop's tensor preservation rule.

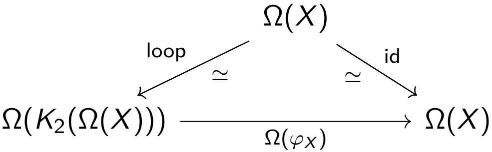
No tricks, but quite large.

The equivalence on objects

We now have $\Omega \circ K_2 \sim \mathrm{id}_{\mathrm{ty}(\mathbf{2Grp})}$. Let's do $K_2 \circ \Omega \sim \mathrm{id}_{\mathrm{ty}(\mathbf{Ptd}^{conn}_{\leq 2})}$.

Let X be a pointed connected 2-type. Define the pointed map $\varphi_X : K_2(\Omega(X)) \to_* X$ by K_2-recursion via the identity 2-group morphism $\Omega(X) \to \Omega(X)$.

By φ_X's computation rule, the following triangle commutes:



Since both X and $K_2(\Omega(X))$ are connected, φ_X is an equivalence.

Bicategories

For us, *bicategory* means $(2, 1)$-category whose 2-cells are paths.

That is, a *bicategory* consists of a type Ob of objects together with

- a doubly indexed family hom of 1-types over Ob
- a composition operation
  $\circ : \mathrm{hom}(b, c) \to \mathrm{hom}(a, b) \to \mathrm{hom}(a, c)$ for all $a, b, c :$ Ob
- an identity morphism id_a for each $a :$ Ob together with two 2-cells (i.e., paths between morphisms): the *right unitor* and the *left unitor*
- an *associator* 2-cell satisfying both the triangle identity with the unitors and the pentagon identity.

Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories.

A *pseudofunctor from $\mathcal{C}$ to $\mathcal{D}$* is a function $F_0 : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ together with

- a function $F_1 : \mathrm{hom}_{\mathcal{C}}(a, b) \to \mathrm{hom}_{\mathcal{D}}(F_0(a), F_0(b))$ for all $a, b : \mathrm{Ob}$
- a 2-cell $F_{\mathrm{id}}(a) : F_1(\mathrm{id}_a) = \mathrm{id}_{F_0(a)}$ for each $a : \mathrm{Ob}$
- a 2-cell $F_\circ(f, g) : F_1(g \circ f) = F_1(g) \circ F_1(f)$ for all composable morphisms f and g
- coherence identities witnessing that $F_\circ$ commutes with the right unitors, with the left unitors, and with the associators.

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be pseudofunctors.

A *pseudotransformation from F to G* consists of

- a component morphism $\eta_0(a) : F_0(a) \to G_0(a)$ for each $a : \mathrm{Ob}(\mathcal{C})$
- a 2-cell $\eta_1(f)$ making the square

$$\begin{array}{ccc} F_0(a) & \xrightarrow{F_1(f)} & F_0(b) \\ {\scriptstyle \eta_0(a)}\big\downarrow & & \big\downarrow{\scriptstyle \eta_0(b)} \\ G_0(a) & \xrightarrow[G_1(f)]{} & G_0(b) \end{array}$$

  commute for all $a, b : \mathrm{Ob}(\mathcal{C})$ and $f : \hom_{\mathcal{C}}(a, b)$.
- a coherence identity witnessing that η_1 commutes with the unitors and one witnessing that it commutes with the associators.

The type of such pseudotransformations is denoted by $F \Rightarrow G$.

The biequivalence

A *biequivalence between* $\mathcal{C}$ *and* $\mathcal{D}$ is a pseudofunctor $F : \mathcal{C} \to \mathcal{D}$ together with

- a pseudofunctor $G : \mathcal{D} \to \mathcal{C}$
- a pseudotransformation $\tau_1 : F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$ each of whose components is an adjoint equivalence in $\mathcal{D}$
- a pseudotransformation $\tau_2 : \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$ each of whose components is an adjoint equivalence in $\mathcal{C}$.

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- a pseudotransformation $\tau_2 : \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$ each of whose components is an adjoint equivalence in $\mathcal{C}$.

Theorem

We have a biequivalence

$$\mathbf{2Grp} \underset{\Omega}{\overset{K_2}{\rightleftarrows}} \mathbf{Ptd}^{conn}_{\leq 2}$$

All data of this biequivalence follow from $K_2(G)$-induction wherever applicable.

An easier way?

From the biequivalence, we extract a pseudofounctor $\Omega : \mathbf{Ptd}^{conn}_{\leq 2} \to \mathbf{2Grp}$ that is an equivalence on objects and homs.

This data ignores most of the hardest constructions for the biequivalence!

By univalence, Ω induces an identity $\mathbf{Ptd}^{conn}_{\leq 2} = \mathbf{2Grp}$.

Then, from an identity, we get whatever we want, including a biequivalence bieq-from-id.

So, why did we construct everything explicitly from scratch?

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So, why did we construct everything explicitly from scratch?

- The full form that bieq-from-id takes is hard to recover.
- Even if recovered, it may be less desirable to work with than our elimination-based constructions.

Future work

1. Can we construct the infinite loop space of a symmetric 2-group G in terms of $K_2(G)$?

   with inspiration from the definition of $K(G, 2)$ from $K(G, 1)$ for abelian groups G

2. Some general bicategory theory inside HoTT:
   - recovering a biequivalence from an isomorphism/identity
   - promoting a biequivalence to an adjoint one.

Conclusion

Takeaway: Coherent 2-groups are biequivalent to pointed connected 2-types.

- **Preprint:**
  `https://phart3.github.io/2Grp-biequiv-preprint.pdf`

- **Agda code:**
  `https://github.com/PHart3/2-groups-agda`

Thanks!

References

John C. Baez, Aaron D. Lauda. 2004. *Higher-Dimensional Algebra V: 2-Groups*.

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Daniel R. Licata, Eric Finster. 2014. *Eilenberg-MacLane spaces in homotopy type theory*.