# Colimits in homotopy type theory

Perry Hart    Kuen-Bang Hou (Favonia)

University of Minnesota, Twin Cities

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# 1 Introduction

This report accompanies our paper "Colimits in homotopy type theory," which is released at https://phart3.github.io/colimits-paper.pdf. It expands on the theorems and proofs provided in the paper. It also contains related material not included in the paper.

### Remarks on notation

- The symbol $=$ denotes the identity type. The symbol $\equiv$ denotes judgmental equality. The symbol $:\equiv$ denotes term definition.
- We use both underbrace and overbrace on terms to indicate $\equiv$ or $:\equiv$.
- For convenience, we'll use the notation

$$\mathsf{PI}(p_1, \ldots, p_n) \ : \ a = b$$

  to denote an equality obtained by simultaneous or iterative path induction on paths $p_1, \ldots, p_n$. We only use this shorthand when the equality is constructed in an evident way.

# 2 Trees

A *(directed) graph* $\Gamma$ is a pair consisting of a type $\Gamma_0 : \mathcal{U}$ and a family $\Gamma_1 : \Gamma_0 \to \Gamma_0 \to \mathcal{U}$ of types.

**Definition 2.0.1.** Let $\Gamma$ be a graph.

1. The *geometric realization* of $\Gamma$ is the coequalizer $|\Gamma| :\equiv {}^{\Gamma_0}/_{\Gamma_1}$, i.e., the HIT generated by the functions
   - $|-| : \Gamma_0 \to {}^{\Gamma_0}/_{\Gamma_1}$
   - $\mathsf{quot\text{-}rel} : \prod_{x,y:\Gamma_0} \Gamma_1(x,y) \to |x| = |y|$.
2. We say that $\Gamma$ is a *tree* if $|\Gamma|$ is contractible.

**Example 2.0.2.**

- Both $\mathbb{N}$ and $\mathbb{Z}$ are trees when viewed as graphs.
- The span $l \leftarrow m \rightarrow r$ is a tree where $l, m, r$ denote the elements of $\mathsf{Fin}(3)$.

**Lemma 2.0.3.** *For every graph* $\Gamma$*,* $\mathsf{colim}_\Gamma\, \mathbf{1} \simeq {}^{\Gamma_0}/_{\Gamma_1}$.

**Corollary 2.0.4.** *For each type* $A$*, define the diagram* $F^A$ *over* $\Gamma$ *by*

$$\begin{aligned} F^A_i &:\equiv A \\ F^A_{i,j,g} &:\equiv \mathsf{id}_A \,. \end{aligned}$$

*If* $\Gamma$ *is a tree, then the function* $[\mathsf{id}_A]_{i:\Gamma_0} : \mathsf{colim}_\Gamma(F^A) \to A$ *is an equivalence.*

*Proof.* Suppose that $\Gamma$ is a tree. Since $A \times -$ has right adjoint $A \to -$, we see that

$$\begin{tikzcd} \mathsf{colim}_\Gamma(F^A) \arrow[r, "\simeq"] \arrow[rrr, bend right, "{[\mathsf{id}_A]}"'] & \mathsf{colim}_\Gamma(F^{(A\times\mathbf{1})}) \arrow[r, "\simeq"] & A \times \mathsf{colim}_\Gamma(F^{\mathbf{1}}) \arrow[r, "\simeq"] & A \end{tikzcd}$$

commutes. $\square$

Let $\Gamma$ be a graph. Define the indexed inductive type of *walks from $i$ to $j$ in $\Gamma$*

$$\begin{aligned} \mathcal{W}_\Gamma &: \Gamma_0 \to \Gamma_0 \to \mathcal{U} \\ \mathsf{nil} &: \prod_{i:\Gamma_0} \mathcal{W}_\Gamma(i,i) \\ \mathsf{cons} &: \prod_{i,j,k:\Gamma_0} \Gamma_1(i,j) \to \mathcal{W}_\Gamma(j,k) \to \mathcal{W}_\Gamma(i,k) \end{aligned}$$

**Definition 2.0.5.** Let $j_0 : \Gamma_0$. We say that $\Gamma$ is a *combinatorial tree* if

- for every $i : \Gamma_0$, we have a term $\nu(i, j_0) : \mathcal{W}_\Gamma(i, j_0)$
- for every $i, j : \Gamma_0$ and $g : \Gamma_1(i,j)$, we have a term

$$\sigma_g \ : \ \nu(i, j_0) = \mathsf{cons}(g, \nu(j, j_0)).$$

**Lemma 2.0.6.** *For every $i, j : \Gamma_0$ and $z : \mathcal{W}_\Gamma(i,j)$, we have a term*

$$\tau(z) \ : \ |i| = |j|\,.$$

*Proof.* We proceed by induction on $\mathcal{W}_\Gamma$.

- Define $\tau(\mathsf{nil}_i) :\equiv \mathsf{refl}_{|i|}$.
- Let $i, j, k : \Gamma_0$, $g : \Gamma_1(i,j)$, and $z : \mathcal{W}_\Gamma(j,k)$. Suppose that $\tau(z)$ has been defined. Define

$$\tau(\mathsf{cons}(g,z)) :\equiv \mathsf{quot\text{-}rel}(g) \cdot \tau(z)$$

$\square$

**Lemma 2.0.7.** *Every combinatorial tree is a tree.*

*Proof.* Let $\Gamma$ be a combinatorial tree. It suffices to prove that for every $x : {}^{\Gamma_0}\!/_{\Gamma_1}$, $x = |j_0|$. We proceed by induction on quotients. For each $i : \Gamma_0$, we have the term

$$\tau(\nu(i, j_0)) \ : \ |i| = |j_0|$$

by Lemma 2.0.6. Since $\Gamma$ is a combinatorial tree, we also see that for all $i, j : \Gamma_0$ and $g : \Gamma_1(i, j)$,

$$\begin{aligned}
&\mathsf{quot\text{-}rel}(g)_*(\tau(\nu(i, j_0))) \\
=\ &\mathsf{quot\text{-}rel}(g)^{-1} \cdot \tau(\nu(i, j_0)) \\
=\ &\mathsf{quot\text{-}rel}(g)^{-1} \cdot \tau(\mathsf{cons}(g, \nu(j, j_0))) \\
\equiv\ &\mathsf{quot\text{-}rel}(g)^{-1} \cdot \mathsf{quot\text{-}rel}(g) \cdot \tau(j, \nu_{j_0,i}) \\
=\ &\tau(j, \nu_{j_0,i})
\end{aligned}$$

This completes the induction proof. □

**Corollary 2.0.8.** *Every directed tree, in the sense of [9, Directed trees], is a tree.*

*Proof.* Just notice that every directed tree is a combinatorial tree. □

**Example 2.0.9.** Trees are abundant in HoTT. Indeed, consider a coalgebra for a polynomial endofunctor $\mathcal{P}_{A,B}$ for a signature $(A, B)$:

$$\mathcal{X} \ :\equiv \ \left(X,\ \alpha : X \to \sum_{a:A} (B(a) \to X)\right)$$

Every element of $X$ has the structure of a directed tree [9, The underlying trees of elements of coalgebras of polynomial endofunctors]. In particular, every element of the $\mathsf{W}$-type for $(A, B)$ is a tree as $\mathsf{W}(A, B)$ has a canonical coalgebra structure [9, W-types as coalgebras for a polynomial endofunctor]. Also, every element of the coinductive type $\mathsf{M}(A, B)$, the terminal coalgebra for $\mathcal{P}_{A,B}$, is a tree.

# 3 Category theory

## 3.1 Wild categories and functors

**Definition 3.1.1.** A *wild category* $\mathcal{C}$ is a tuple consisting of

$$\begin{aligned}
\mathsf{Ob} &: \mathcal{U} \\
\mathsf{hom}_{\mathcal{C}} &: C \to C \to \mathcal{U} \\
\circ &: \prod_{X,Y,Z:C} \mathsf{hom}_{\mathcal{C}}(Y,Z) \to \mathsf{hom}_{\mathcal{C}}(X,Y) \to \mathsf{hom}_{\mathcal{C}}(X,Z) \\
\mathsf{id} &: \prod_{X:C} \mathsf{hom}_{\mathcal{C}}(X,X) \\
\mathsf{RId} &: \prod_{X,Y:C} \prod_{f:\mathsf{hom}_{\mathcal{C}}(X,Y)} f \circ \mathsf{id}_X = f \\
\mathsf{LId} &: \prod_{X,Y:C} \prod_{f:\mathsf{hom}_{\mathcal{C}}(X,Y)} \mathsf{id}_Y \circ f = f \\
\mathsf{assoc} &: \prod_{W,X,Y,Z:C} \prod_{h:\mathsf{hom}_{\mathcal{C}}(Y,Z)} \prod_{g:\mathsf{hom}_{\mathcal{C}}(X,Y)} \prod_{f:\mathsf{hom}_{\mathcal{C}}(W,X)} (h \circ g) \circ f = h \circ (g \circ f)
\end{aligned}$$

**Definition 3.1.2.** A functor $F : \mathcal{C} \to \mathcal{D}$ between wild categories consists of

$$\begin{aligned}
F_0 &: \mathsf{Ob}(\mathcal{C}) \to \mathsf{Ob}(\mathcal{D}) \\
F_1 &: \prod_{X,Y:\mathsf{Ob}(\mathcal{C})} \mathsf{hom}_{\mathcal{C}}(X,Y) \to \mathsf{hom}_{\mathcal{D}}(F_0(X), F_0(Y)) \\
F_2 &: \prod_{X,Y,Z:\mathsf{Ob}(\mathcal{C})} \prod_{g:\mathsf{hom}_{\mathcal{C}}(Y,Z)} \prod_{f:\mathsf{hom}_{\mathcal{C}}(X,Y)} F_1(g \circ f) = F_1(g) \circ F_1(f) \\
F_3 &: \prod_{X:\mathsf{Ob}(\mathcal{C})} F_1(\mathsf{id}_X) = \mathsf{id}_{F_0(X)}
\end{aligned}$$

We may refer to $F_0$ or $F_1$ by $F$. Also, if the data $F_2$ and $F_3$ are omitted, then we call $F$ a 0-*functor*.

**Definition 3.1.3.** Let $\mathcal{C}$ be a wild category. A *reflective subuniverse* of $\mathcal{C}$ is a predicate $P : \mathsf{Ob}(\mathcal{C}) \to \mathsf{Prop}$ together with functions

$$\begin{aligned}
\bigcirc &: \mathsf{Ob}(\mathcal{C}) \to \mathsf{Ob}(\mathcal{C}) \\
\eta &: \prod_{X:\mathsf{Ob}(\mathcal{C})} \mathsf{hom}_{\mathcal{C}}(X, \bigcirc X)
\end{aligned}$$

such that

- for each $X : \mathsf{Ob}(\mathcal{C})$, $P(\bigcirc X)$
- for each $X, Y : \mathsf{Ob}(\mathcal{C})$ with $P(Y)$, the function

$$\mathsf{hom}_{\mathcal{C}}(\bigcirc X, Y) \xrightarrow{f \mapsto f \circ \eta_X} \mathsf{hom}_{\mathcal{C}}(X, Y)$$

is an equivalence. The inverse of this map is denoted by $\mathsf{rec}_{\bigcirc}$.

We define $\mathcal{C}_P :\equiv \sum_{X:\mathsf{Ob}(\mathcal{C})} P(X)$.

**Proposition 3.1.4.** *For each $X : \mathsf{Ob}(\mathcal{C})$, the square*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\
\bigcirc X \arrow[r, "\mathsf{rec}_{\bigcirc}(\eta_Y \circ f)"'] & \bigcirc Y
\end{tikzcd}$$

*in $\mathcal{C}$ commutes.*

**Definition 3.1.5.** A *bicategory* is a wild category $\mathcal{C}$ equipped with

- an identity

$$\begin{array}{c}
\mathsf{ap}_{-\circ f}(\mathsf{assoc}(k,g,h)) \cdot \mathsf{assoc}(k, g\circ h, f) \cdot \mathsf{ap}_{k\circ -}(\mathsf{assoc}(g,h,k)) \\
\Big\Vert \rho(f,k,g,h) \\
\mathsf{assoc}(k\circ g, h, f) \cdot \mathsf{assoc}(k,g,h\circ f)
\end{array}$$

  for all composable morphism $k$, $g$, $h$, and $f$.

- an identity

$$\begin{array}{c}
\mathsf{assoc}(g, \mathsf{id}, h) \cdot \mathsf{ap}_{g\circ -}(\mathsf{Lid}(h)) \\
\Big\Vert \upsilon(g,h) \\
\mathsf{ap}_{-\circ h}(\mathsf{Rid}(g))
\end{array}$$

  for all composable morphisms $g$ and $h$.

**Lemma 3.1.6.** *Let $\mathcal{C}$ be a bicategory. For every $A, B, C : \mathsf{Ob}(\mathcal{C})$, $f : \mathsf{hom}_{\mathcal{C}}(A,B)$, $g : \mathsf{hom}_{\mathcal{C}}(B,C)$, we have that*

$$\mathsf{Lid}(g\circ f)^{-1} \cdot \mathsf{assoc}(\mathsf{id}, g, f)^{-1} \cdot \mathsf{ap}_{-\circ f}(\mathsf{Lid}(g)) \;=\; \mathsf{refl}_{g\circ f}.$$

*Proof.* Since the function $(c = d) \xrightarrow{\mathsf{ap}_{\mathsf{id}\,\circ -}} (\mathsf{id}\circ c = \mathsf{id}\circ d)$ has a retraction for all parallel morphisms $c$ and $d$, it suffices to prove that

$$\mathsf{ap}_{\mathsf{id}\,\circ -}(\mathsf{Lid}(g\circ f))^{-1} \cdot \mathsf{ap}_{\mathsf{id}\,\circ -}(\mathsf{assoc}(\mathsf{id}, g, f))^{-1} \cdot \mathsf{ap}_{\mathsf{id}\,\circ -}(\mathsf{ap}_{-\circ f}(\mathsf{Lid}(g))) \;=\; \mathsf{refl}_{\mathsf{id}\,\circ(g\circ f)}.$$

Note that the left two subdiagrams of

$$\begin{tikzcd}
((\mathsf{id}\circ\mathsf{id})\circ g)\circ f \arrow[r, equal, "\mathsf{ap}_{-\circ f}(\mathsf{assoc}(\mathsf{id},\mathsf{id},g))"] \arrow[dr, equal, "\mathsf{ap}_{-\circ f}(\mathsf{ap}_{-\circ g}(\mathsf{RId}(\mathsf{id})))"'] & (\mathsf{id}\circ(\mathsf{id}\circ g))\circ f \arrow[r, equal, "\mathsf{assoc}(\mathsf{id},\mathsf{id}\circ g,f)"] \arrow[d, equal, "\mathsf{ap}_{-\circ f}(\mathsf{ap}_{\mathsf{id}\,\circ -}(\mathsf{LId}(g)))"] & \mathsf{id}\circ((\mathsf{id}\circ g)\circ f) \arrow[r, equal, "\mathsf{ap}_{\mathsf{id}\,\circ -}(\mathsf{assoc}(\mathsf{id},g,f))"] \arrow[d, equal, "\mathsf{ap}_{\mathsf{id}\,\circ -}(\mathsf{ap}_{-\circ f}(\mathsf{LId}(g)))"] & \mathsf{id}\circ(\mathsf{id}\circ(g\circ f)) \arrow[dl, equal, "\mathsf{ap}_{\mathsf{id}\,\circ -}(\mathsf{LId}(g\circ f))"] \\
 & (\mathsf{id}\circ g)\circ f \arrow[r, equal, "\mathsf{assoc}(\mathsf{id},g,f)"'] & \mathsf{id}\circ(g\circ f) &
\end{tikzcd}$$

commute, and we want to prove that the right one commutes. Hence it suffices to prove that this diagram's outer perimeter commutes. This follows from the commuting diagram

$$\begin{tikzcd}
 & (\mathsf{id} \circ (\mathsf{id} \circ g)) \circ f \arrow[r, equal, "\mathsf{assoc}(\mathsf{id}, \mathsf{id} \circ g, f)"] & \mathsf{id} \circ ((\mathsf{id} \circ g) \circ f) \arrow[d, equal, "\mathsf{ap}_{\mathsf{id} \circ -}(\mathsf{assoc}(\mathsf{id}, g, f))"] \\
((\mathsf{id} \circ \mathsf{id}) \circ g) \circ f \arrow[ur, equal, bend left, "\mathsf{ap}_{- \circ f}(\mathsf{assoc}(\mathsf{id}, \mathsf{id}, g))"] \arrow[r, equal, "\mathsf{assoc}(\mathsf{id} \circ \mathsf{id}, g, f)"'] \arrow[d, equal, "\mathsf{ap}_{- \circ f}(\mathsf{ap}_{- \circ g}(\mathsf{RId}(\mathsf{id})))"'] & (\mathsf{id} \circ \mathsf{id}) \circ (g \circ f) \arrow[r, equal, "\mathsf{assoc}(\mathsf{id}, \mathsf{id}, g \circ f)"'] \arrow[d, equal, "\mathsf{ap}_{- \circ (g \circ f)}(\mathsf{RId}(\mathsf{id}))"] & \mathsf{id} \circ (\mathsf{id} \circ (g \circ f)) \arrow[dl, equal, bend left, "\mathsf{ap}_{\mathsf{id} \circ -}(\mathsf{LId}(g \circ f))"] \\
(\mathsf{id} \circ g) \circ f \arrow[r, equal, "\mathsf{assoc}(\mathsf{id}, g, f)"'] & \mathsf{id} \circ (g \circ f) &
\end{tikzcd}$$

□

**Definition 3.1.7.** Let $\mathcal{C}$ be a wild category.

- A morphism $f : \mathsf{hom}_{\mathcal{C}}(A, B)$ of $\mathcal{C}$ is an *equivalence* if it is biinvertible:

$$\mathsf{is\_equiv}(f) \;:=\; \sum_{g,h:\mathsf{hom}_{\mathcal{C}}(B,A)} g \circ f = \mathsf{id}_A \times f \circ h = \mathsf{id}_B$$

  We write $A \simeq_{\mathcal{C}} B$ for the type of equivalences from $A$ to $B$.

- We say that $\mathcal{C}$ is *univalent* if the function

$$\begin{aligned} \mathsf{idtoequiv} \;&:\; \big(A =_{\mathsf{Ob}(\mathcal{C})} B\big) \to (A \simeq_{\mathcal{C}} B) \\ \mathsf{refl}_A \;&\mapsto\; (\mathsf{id}_A, \mathsf{id}_A, \mathsf{id}_A, \mathsf{LId}(\mathsf{id}_A), \mathsf{LId}(\mathsf{id}_A)) \end{aligned}$$

  is an equivalence.

**Example 3.1.8.** The category $\mathcal{U}$ of types and functions is a bicategory and (assuming the univalence axiom) is univalent.

**Lemma 3.1.9.** *Suppose that $\mathcal{C}$ is a wild category. Let $(P, \bigcirc, \eta)$ be a reflective subuniverse on $\mathcal{C}$. For each $X : \mathsf{Ob}(\mathcal{C})$, $P(X) \to \mathsf{is\_equiv}(\eta_X)$.*

*Proof.* Let $X : \mathsf{Ob}(\mathcal{C})$. The type $T_{P,X}$ of tuples

$$\begin{aligned} Y \;&:\; \mathsf{Ob}(\mathcal{C}) \\ q \;&:\; P(Y) \\ f \;&:\; \mathsf{hom}_{\mathcal{C}}(X, Y) \\ I \;&:\; \prod_{Z:\mathsf{Ob}(\mathcal{C})} P(Z) \to \mathsf{is\_equiv}(\lambda(g : \mathsf{hom}_{\mathcal{C}}(Y, Z)).g \circ f) \end{aligned}$$

is a mere proposition. Now, suppose that $P(X)$. We have terms

$$(X, \ldots, \mathsf{id}_X, \ldots)$$
$$(\bigcirc X, \ldots, \eta_X, \ldots)$$

of type $T_{P,X}$, which must be equal. Therefore, we have a commuting triangle

$$\begin{tikzcd} & X \arrow[dl, "\mathsf{id}"'] \arrow[dr, "\eta_X"] & \\ X \arrow[rr, dashed, "\simeq"'] & & \bigcirc X \end{tikzcd}$$

in $\mathcal{C}$. This implies that $\eta_X$ is an equivalence. □

By Proposition 3.1.4, it follows that $\eta$ restricted to $\mathcal{U}_P$ is a *natural* isomorphsm $\mathsf{id}_{\mathcal{U}_P} \xrightarrow{\simeq} \bigcirc \circ \mathcal{I}$ of functors where $\mathcal{I}$ denotes the inclusion of the subuniverse $\mathcal{U}_P$ into $\mathcal{U}$.

## 3.2 Orthogonal factorization systems

**Definition 3.2.1.** Let $\mathcal{C}$ be a wild category. An *orthogonal factorization system (OFS)* on $\mathcal{C}$ consists of predicates

$$\mathcal{L}, \mathcal{R} \;:\; \prod_{A,B \,:\, \mathcal{C}} \mathsf{hom}_{\mathcal{C}}(A, B) \to \mathsf{Prop}$$

such that

1. both $\mathcal{L}$ and $\mathcal{R}$ are closed under composition and have all identities;

2. for every $h : \mathsf{hom}_{\mathcal{C}}(A, B)$, the type

$$\mathsf{fact}_{\mathcal{L},\mathcal{R}}(h) \;\coloneqq\; \sum_{D:\mathcal{C}} \sum_{f:\mathsf{hom}_{\mathcal{C}}(A,D)} \sum_{g:\mathsf{hom}_{\mathcal{C}}(D,B)} (g \circ f) = h \times \mathcal{L}(f) \times \mathcal{R}(g)$$

   is contractible.

**Example 3.2.2.** Rijke et al. use a particular indexed recursive 1-HIT to show that every family $\prod_{a:A} F(a) \to G(a)$ of functions induces an OFS on $\mathcal{U}$ [8, Section 2.4].

**Definition 3.2.3 (Lifting property).** Let $\mathcal{C}$ be a wild category. Let $l : \mathsf{hom}_{\mathcal{C}}(A, B)$ and $\mathcal{H}$ be a property of morphisms in $\mathcal{C}$. We say that $l$ has the *left lifting property against* $\mathcal{H}$ if for every $r : \mathsf{hom}_{\mathcal{C}}(C, D)$ with $r \in \mathcal{H}$ and every commuting square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] \arrow[dr, phantom, "S"] & C \arrow[d, "r"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

the type

$$\mathsf{fill}(S) \;:=\; \sum_{d:\mathsf{hom}_{\mathcal{C}}(B,C)} \sum_{H_f:d\circ l=f} \sum_{H_g:r\circ d=g} \mathsf{assoc}(r,d,l)\cdot \mathsf{ap}_{r\circ -}(H_f) = \mathsf{ap}_{-\circ l}(H_g)\cdot S$$

of *diagonal fillers* is contractible. In this case, we write ${}^{\perp}\mathcal{H}(l)$.

The predicate *right lifting property* is defined similarly.

Let $\mathcal{C}$ be a univalent bicategory and $(\mathcal{L},\mathcal{R})$ be an OFS on $\mathcal{C}$.

**Lemma 3.2.4.** *Let $h : \mathsf{hom}_{\mathcal{C}}(N,M)$ and $(U,s_U,t_U,p_U),(V,s_V,t_V,p_V) : \mathsf{fact}_{\mathcal{L},\mathcal{R}}(h)$. We have that*

$$((U,s_U,t_U,p_U)=(V,s_V,t_V,p_V)) \;\simeq\; \textstyle\sum_{e:U\simeq_{\mathcal{C}}V} \underbrace{\displaystyle\sum_{H_{\mathcal{L}}:s_V=e\circ s_U} \sum_{H_{\mathcal{R}}:t_U=t_V\circ e} \mathsf{ap}_{t_V\circ -}(H_{\mathcal{L}})\cdot \mathsf{assoc}(t_V,e,s_U)^{-1}\cdot \mathsf{ap}_{-\circ s_U}(H_{\mathcal{R}})^{-1}\cdot p_U = p_V}_{Y(V,s_V,t_V,p_V,e)}$$

*Proof.* For each $U : \mathsf{Ob}(\mathcal{C})$, the type family $V \mapsto U \simeq_{\mathcal{C}} V$ is an identity system on $(\mathsf{Ob}(\mathcal{C}),U)$ because $\mathcal{C}$ is univalent. We also have the type family

$$D \;\mapsto\; \sum_{f:\mathsf{hom}(A,D)} \sum_{g:\mathsf{hom}(D,B)} g\circ f = h$$

over $\mathsf{Ob}(\mathcal{C})$, which is pointed over $U$ at $(s_U,t_U,p_U)$. Now, we have a term

$$\left(\mathsf{LId}(s_U)^{-1},\mathsf{RId}(t_U)^{-1},\hat{v}(t_U,s_U)\right) \;:\; Y(U,s_U,t_U,p_U,\mathsf{id}_U)$$

where $\hat{v}(t_U,s_U)$ comes from Definition 3.1.5. It follows that $Y(V,s_V,t_V,p_V,e)$ is an SNS on $\left(\sum_{f:\mathsf{hom}(A,D)}\sum_{g:\mathsf{hom}(D,B)} g\circ f = h, U\simeq_{\mathcal{C}} V\right)$. Note that

$$\begin{gathered}
\textstyle\sum_{s_V:\mathsf{hom}(N,U)}\sum_{t_V:\mathsf{hom}(U,M)}\sum_{p_V:t_V\circ s_V=h}\sum_{H_{\mathcal{L}}:s_V=\mathsf{id}_U\circ s_U}\sum_{H_{\mathcal{R}}:t_U=t_V\circ\mathsf{id}_U} \mathsf{ap}_{t_V\circ -}(H_{\mathcal{L}})\cdot\mathsf{assoc}(t_V,\mathsf{id}_U,s_U)^{-1}\cdot\mathsf{ap}_{-\circ s_U}(H_{\mathcal{R}})^{-1}\cdot p_U = p_V\\
\|\\
\textstyle\sum_{s_V:\mathsf{hom}(N,U)}\sum_{H_{\mathcal{L}}:s_V=\mathsf{id}_U\circ s_U}\sum_{t_V:\mathsf{hom}(U,M)}\sum_{H_{\mathcal{R}}:t_U=t_V\circ\mathsf{id}_U}\sum_{p_V:t_V\circ s_V=h} \mathsf{ap}_{t_V\circ -}(H_{\mathcal{L}})\cdot\mathsf{assoc}(t_V,\mathsf{id}_U,s_U)^{-1}\cdot\mathsf{ap}_{-\circ s_U}(H_{\mathcal{R}})^{-1}\cdot p_U = p_V\\
\|\\
\textstyle\sum_{t_V:\mathsf{hom}(U,M)}\sum_{H_{\mathcal{R}}:t_U=t_V\circ\mathsf{id}_U}\sum_{p_V:t_V\circ(\mathsf{id}_U\circ s_U)=h} \mathsf{assoc}(t_V,\mathsf{id}_U,s_U)^{-1}\cdot\mathsf{ap}_{-\circ s_U}(H_{\mathcal{R}})^{-1}\cdot p_U = p_V\\
\|\\
\textstyle\sum_{t_V:\mathsf{hom}(U,M)}\sum_{H_{\mathcal{R}}:t_U=t_V}\sum_{p_V:t_V\circ(\mathsf{id}_U\circ s_U)=h} \mathsf{assoc}(t_V,\mathsf{id}_U,s_U)^{-1}\cdot\mathsf{ap}_{-\circ s_U}(H_{\mathcal{R}}\cdot\mathsf{RId}(t_V)^{-1})^{-1}\cdot p_U = p_V\\
\|\\
\textstyle\sum_{p_V:t_U\circ(\mathsf{id}_U\circ s_U)=h} \mathsf{assoc}(t_U,\mathsf{id}_U,s_U)^{-1}\cdot\mathsf{ap}_{-\circ s_U}(\mathsf{RId}(t_U)^{-1})^{-1}\cdot p_U = p_V\\
\|\\
\mathbf{1}
\end{gathered}$$

Therefore, by Theorem A.0.3, we have our desired equivalence. □

**Lemma 3.2.5 (Unique lifting property).** *For each commuting square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] \arrow[dr, phantom, "S"] & X \arrow[d, "r"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

*such that $l \in \mathcal{L}$ and $r \in \mathcal{R}$, the type $\mathsf{fill}(S)$ of diagonal fillers is contractible.*

*Proof.* Note that we have a commuting diagram

$$\begin{tikzcd} A \arrow[r, "s_f"] \arrow[rr, bend left, "f"] \arrow[rr, phantom, bend left=15, "p_f"] \arrow[d, "l"'] & \mathrm{im}(f) \arrow[r, "t_f"] & X \arrow[d, "r"] \\ B \arrow[r, "s_g"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, bend right=15, "p_g"'] & \mathrm{im}(g) \arrow[r, "t_g"] & Y \end{tikzcd}$$

Since $\mathsf{fact}_{\mathcal{L},\mathcal{R}}(r \circ f)$ is contractible, so is its identity type

$$\big(\mathrm{im}(f), s_f, r \circ t_f, \mathsf{assoc}(r, t_f, s_f) \cdot \mathsf{ap}_{r\circ-}(p_f)\big) \;=\; \big(\mathrm{im}(g), s_g \circ l, t_g, \mathsf{assoc}(t_g, s_g, l)^{-1} \cdot \mathsf{ap}_{-\circ l}(p_g) \cdot S\big)$$

By Lemma 3.2.4, the type

$$\textstyle\sum_{e:\mathrm{im}(f)\simeq_{\mathcal{C}}\mathrm{im}(g)} \sum_{H_{\mathcal{L}}:s_g\circ l=e\circ s_f} \sum_{H_{\mathcal{R}}:r\circ t_f=t_g\circ e} \mathsf{ap}_{t_g\circ-}(H_{\mathcal{L}}) \cdot \mathsf{assoc}(t_g, e, s_f)^{-1} \cdot \mathsf{ap}_{-\circ s_f}(H_{\mathcal{R}})^{-1} \cdot \mathsf{assoc}(r, t_f, s_f) \cdot \mathsf{ap}_{r\circ-}(p_f) = \mathsf{assoc}(t_g, s_g, l)^{-1} \cdot \mathsf{ap}_{-\circ l}(p_g) \cdot S$$

is also contractible. Moreover, we have the chain of equalities shown on the next page.

$$\textstyle\sum_{e:\mathrm{im}(f)\simeq_{\mathcal{C}}\mathrm{im}(g)}\sum_{H_{\mathcal{L}}:s_g\circ l=e\circ s_f}\sum_{H_{\mathcal{R}}:r\circ t_f=t_g\circ e}\mathsf{ap}_{t_g\circ-}(H_{\mathcal{L}})\cdot\mathsf{assoc}(t_g,e,s_f)^{-1}\cdot\mathsf{ap}_{-\circ s_f}(H_{\mathcal{R}})^{-1}\cdot\mathsf{assoc}(r,t_f,s_f)\cdot\mathsf{ap}_{r\circ-}(p_f)=\mathsf{assoc}(t_g,s_g,l)^{-1}\cdot\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\Vert$$

$$\textstyle\sum_{I:\mathsf{Ob}(\mathcal{C})}\sum_{a_1:\mathsf{hom}_{\mathcal{C}}(A,I)}\sum_{a_2:\mathsf{hom}_{\mathcal{C}}(I,X)}\sum_{p_f:a_2\circ a_1=f}\sum_{b_1:\mathsf{hom}_{\mathcal{C}}(B,I)}\sum_{b_2:\mathsf{hom}_{\mathcal{C}}(I,Y)}\sum_{p_g:b_2\circ b_1=g}\sum_{H_{\mathcal{L}}:b_1\circ l=\mathsf{id}\circ a_1}\sum_{H_{\mathcal{R}}:r\circ a_2=b_2\circ\mathsf{id}}\sum_{a_1,b_1\in\mathcal{L}}\sum_{a_2,b_2\in\mathcal{R}}\mathsf{ap}_{b_2\circ-}(H_{\mathcal{L}})\cdot\mathsf{assoc}(b_2,\mathsf{id},a_1)^{-1}\cdot\mathsf{ap}_{-\circ a_1}(H_{\mathcal{R}})^{-1}\cdot\mathsf{assoc}(r,a_2,a_1)\cdot\mathsf{ap}_{r\circ-}(p_f)=\mathsf{assoc}(b_2,b_1,l)^{-1}\cdot\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\Vert$$

$$\textstyle\sum_{I:\mathsf{Ob}(\mathcal{C})}\sum_{a_1:\mathsf{hom}_{\mathcal{C}}(A,I)}\sum_{a_2:\mathsf{hom}_{\mathcal{C}}(I,X)}\sum_{p_f:a_2\circ a_1=f}\sum_{b_1:\mathsf{hom}_{\mathcal{C}}(B,I)}\sum_{b_2:\mathsf{hom}_{\mathcal{C}}(I,Y)}\sum_{p_g:b_2\circ b_1=g}\sum_{H_{\mathcal{L}}:b_1\circ l=a_1}\sum_{H_{\mathcal{R}}:r\circ a_2=b_2}\sum_{a_1,b_1\in\mathcal{L}}\sum_{a_2,b_2\in\mathcal{R}}\mathsf{ap}_{b_2\circ-}(H_{\mathcal{L}})\cdot\mathsf{ap}_{b_2\circ-}(\mathsf{Lid}(a_1))^{-1}\cdot\mathsf{assoc}(b_2,\mathsf{id},a_1)^{-1}\cdot\mathsf{ap}_{-\circ a_1}(\mathsf{Rid}(b_2))\cdot\mathsf{ap}_{-\circ a_1}(H_{\mathcal{R}})^{-1}\cdot\mathsf{assoc}(r,a_2,a_1)\cdot\mathsf{ap}_{r\circ-}(p_f)=\mathsf{assoc}(b_2,a_2,l)^{-1}\cdot\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\Vert$$

$$\textstyle\sum_{I:\mathsf{Ob}(\mathcal{C})}\sum_{a_2:\mathsf{hom}_{\mathcal{C}}(I,X)}\sum_{p_f:a_2\circ(b_1\circ l)=f}\sum_{b_1:\mathsf{hom}_{\mathcal{C}}(B,I)}\sum_{p_g:(r\circ a_2)\circ b_1=g}\sum_{b_1\in\mathcal{L}}\sum_{a_2\in\mathcal{R}}\mathsf{ap}_{b_2\circ-}(\mathsf{Lid}(b_1\circ l))^{-1}\cdot\mathsf{assoc}(b_2,\mathsf{id},a_1)^{-1}\cdot\mathsf{ap}_{-\circ a_1}(\mathsf{Rid}(r\circ a_2))\cdot\mathsf{assoc}(r,a_2,a_1)\cdot\mathsf{ap}_{r\circ-}(p_f)=\mathsf{assoc}(b_2,b_1,l)^{-1}\cdot\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\Vert$$

$$\textstyle\sum_{I:\mathsf{Ob}(\mathcal{C})}\sum_{a_2:\mathsf{hom}_{\mathcal{C}}(I,X)}\sum_{b_1:\mathsf{hom}_{\mathcal{C}}(B,I)}\sum_{p_f:(a_2\circ b_1)\circ l=f}\sum_{p_g:r\circ(a_2\circ b_1)=g}\sum_{b_1\in\mathcal{L}}\sum_{a_2\in\mathcal{R}}\mathsf{ap}_{(r\circ a_2)\circ-}(\mathsf{Lid}(b_1\circ l))^{-1}\cdot\mathsf{assoc}(r\circ a_2,\mathsf{id},b_1\circ l)^{-1}\cdot\mathsf{ap}_{-\circ(b_1\circ l)}(\mathsf{Rid}(r\circ a_2))\cdot\mathsf{assoc}(r,a_2,b_1\circ l)\cdot\mathsf{ap}_{r\circ-}(\mathsf{assoc}(a_2,b_1,l))^{-1}\cdot\mathsf{ap}_{r\circ-}(p_f)=\mathsf{assoc}(r\circ a_2,b_1,l)^{-1}\cdot\mathsf{ap}_{-\circ l}(\mathsf{assoc}(r,a_2,b_1))\cdot\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\Vert$$

$$\textstyle\sum_{I:\mathsf{Ob}(\mathcal{C})}\sum_{a_2:\mathsf{hom}_{\mathcal{C}}(I,X)}\sum_{b_1:\mathsf{hom}_{\mathcal{C}}(B,I)}\sum_{p_f:(a_2\circ b_1)\circ l=f}\sum_{p_g:r\circ(a_2\circ b_1)=g}\sum_{b_1\in\mathcal{L}}\sum_{a_2\in\mathcal{R}}\mathsf{ap}_{-\circ l}(\mathsf{assoc}(r,a_2,b_1))^{-1}\cdot\mathsf{assoc}(r\circ a_2,b_1,l)\cdot\overbrace{\mathsf{ap}_{(r\circ a_2)\circ-}(\mathsf{Lid}(b_1\circ l))^{-1}\cdot\mathsf{assoc}(r\circ a_2,\mathsf{id},b_1\circ l)^{-1}\cdot\mathsf{ap}_{-\circ(b_1\circ l)}(\mathsf{Rid}(r\circ a_2))}^{\mathsf{refl}_{(r\circ a_2)\circ(b_1\circ l)}}\cdot\mathsf{assoc}(r,a_2,b_1\circ l)\cdot\mathsf{ap}_{r\circ-}(\mathsf{assoc}(a_2,b_1,l))^{-1}\cdot\mathsf{ap}_{r\circ-}(p_f)=\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\Vert$$

$$\textstyle\sum_{d:\mathsf{hom}_{\mathcal{C}}(B,X)}\underbrace{\displaystyle\sum_{I:\mathsf{Ob}(\mathcal{C})}\sum_{a_2:\mathsf{hom}_{\mathcal{C}}(I,X)}\sum_{b_1:\mathsf{hom}_{\mathcal{C}}(B,I)}\sum_{b_1\in\mathcal{L}}\sum_{a_2\in\mathcal{R}}\sum_{H_j:a_2\circ b_1=d}}_{\mathsf{fact}_{\mathcal{L},\mathcal{R}}(d)}\textstyle\sum_{p_f:(a_2\circ b_1)\circ l=f}\sum_{p_g:r\circ(a_2\circ b_1)=g}\mathsf{assoc}(r,a_2\circ b_1,l)\cdot\mathsf{ap}_{r\circ-}(p_f)=\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\Vert$$

$$\mathsf{fill}(S)$$

It follows that $\mathsf{fill}(S)$ is contractible. $\square$

**Corollary 3.2.6.** *We have that $\mathcal{L} = {}^{\perp}\mathcal{R}$ and $\mathcal{L}^{\perp} = \mathcal{R}$.*

*Proof.* We just prove that $\mathcal{L} = {}^{\perp}\mathcal{R}$ as the other case is dual. Let $f : \mathsf{hom}_{\mathcal{C}}(A, B)$. By Lemma 3.2.5, we know that $\mathcal{L}(f) \to {}^{\perp}\mathcal{R}(f)$. To prove the reverse implication, suppose that ${}^{\perp}\mathcal{R}(f)$. Factor $f$ as $(\mathrm{im}(f), s_f, t_f, p_f)$ and consider the commuting square

$$\begin{tikzcd}
A \arrow[r, "s_f"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{Lid}(f)\cdot p_f^{-1}"] & \mathrm{im}(f) \arrow[d, "t_f"] \\
B \arrow[r, "\mathsf{id}"'] & B
\end{tikzcd}$$

Since ${}^{\perp}\mathcal{R}(f)$, the type $\mathsf{fill}(\mathsf{Lid}(f) \cdot p_f^{-1})$ is contractible, with center, say, $\big(d, H_{s_f}, H_{\mathsf{id}}, K\big)$. Now, the commuting square

$$\begin{tikzcd}
A \arrow[r, "s_f"] \arrow[d, "s_f"'] \arrow[dr, phantom, "\mathsf{refl}_{t_f \circ s_f}"] & \mathrm{im}(f) \arrow[d, "t_f"] \\
\mathrm{im}(f) \arrow[r, "t_f"'] & B
\end{tikzcd}$$

is contractible and has two diagonal fillers:

$$(\mathsf{id}, \mathsf{Lid}(s_f), \mathsf{Rid}(t_f), \dots)$$
$$\Big(d \circ t_f, \mathsf{assoc}(d, t_f, s_f) \cdot \mathsf{ap}_{d\circ -}(p_f) \cdot H_{s_f}, \mathsf{assoc}(t_f, d, t_f)^{-1} \cdot \mathsf{ap}_{-\circ t_f}(H_{\mathsf{id}}) \cdot \mathsf{Lid}(t_f), \omega\Big)$$

where $\omega$ denotes the chain of equalities shown on the next page.

$$
\begin{gathered}
\mathsf{assoc}(t_f, d\circ t_f, s_f)\cdot \mathsf{ap}_{t_f\circ -}(\mathsf{assoc}(d, t_f, s_f)\cdot \mathsf{ap}_{d\circ -}(p_f)\cdot H_{s_f})\\
\Big\Vert\\
\mathsf{assoc}(t_f, d\circ t_f, s_f)\cdot \mathsf{ap}_{t_f\circ -}(\mathsf{assoc}(d, t_f, s_f))\cdot \mathsf{ap}_{t_f\circ(d\circ -)}(p_f)\cdot \mathsf{ap}_{t_f\circ -}(H_{s_f})\\
\Big\Vert\, {\scriptstyle via\ K}\\
\mathsf{assoc}(t_f, d\circ t_f, s_f)\cdot \mathsf{ap}_{t_f\circ -}(\mathsf{assoc}(d, t_f, s_f))\cdot \mathsf{ap}_{t_f\circ(d\circ -)}(p_f)\cdot \mathsf{assoc}(t_f, d, f)^{-1}\cdot \mathsf{ap}_{-\circ f}(H_{\mathsf{id}})\cdot \mathsf{Lid}(f)\cdot p_f^{-1}\\
\Big\Vert\\
\mathsf{ap}_{-\circ s_f}(\mathsf{assoc}(t_f, d, t_f))^{-1}\cdot \mathsf{assoc}(t_f\circ d, t_f, s_f)\cdot \mathsf{assoc}(t_f, d, t_f\circ s_f)\cdot \mathsf{ap}_{t_f\circ(d\circ -)}(p_f)\cdot \mathsf{assoc}(t_f, d, f)^{-1}\cdot \mathsf{ap}_{-\circ f}(H_{\mathsf{id}})\cdot \mathsf{Lid}(f)\cdot p_f^{-1}\cdot \mathsf{Lid}(t_f\circ s_f)^{-1}\cdot \mathsf{assoc}(\mathsf{id}, t_f, s_f)^{-1}\cdot \mathsf{ap}_{-\circ s_f}(\mathsf{Lid}(t_f))\\
\Big\Vert\\
\mathsf{ap}_{-\circ s_f}(\mathsf{assoc}(t_f, d, t_f))^{-1}\cdot \mathsf{assoc}(t_f\circ d, t_f, s_f)\cdot \mathsf{ap}_{(t_f\circ d)\circ -}(p_f)\cdot \mathsf{ap}_{-\circ f}(H_{\mathsf{id}})\cdot \mathsf{ap}_{\mathsf{id}\circ -}(p_f)^{-1}\cdot \mathsf{assoc}(\mathsf{id}, t_f, s_f)^{-1}\cdot \mathsf{ap}_{-\circ s_f}(\mathsf{Lid}(t_f))\\
\Big\Vert\\
\mathsf{ap}_{-\circ s_f}(\mathsf{assoc}(t_f, d, t_f))^{-1}\cdot \mathsf{assoc}(t_f\circ d, t_f, s_f)\cdot \mathsf{ap}_{-\circ(t_f\circ s_f)}(H_{\mathsf{id}})\cdot \mathsf{assoc}(\mathsf{id}, t_f, s_f)^{-1}\cdot \mathsf{ap}_{-\circ s_f}(\mathsf{Lid}(t_f))\\
\Big\Vert\\
\mathsf{ap}_{-\circ s_f}(\mathsf{assoc}(t_f, d, t_f))^{-1}\cdot \mathsf{ap}_{(-\circ t_f)\circ s_f}(H_{\mathsf{id}})\cdot \mathsf{ap}_{-\circ s_f}(\mathsf{Lid}(t_f))\\
\Big\Vert\\
\mathsf{ap}_{-\circ s_f}(\mathsf{assoc}(t_f, d, t_f)^{-1}\cdot \mathsf{ap}_{-\circ t_f}(H_{\mathsf{id}})\cdot \mathsf{Lid}(t_f))
\end{gathered}
$$

It follows that $d$ is an equivalence with inverse $t_f$. Thus, $d \in \mathcal{L}$. $\square$

**Lemma 3.2.7.** *Let $(\mathcal{L}, \mathcal{R})$ be an OFS on the category $\mathcal{U}$ of types. Consider a pushout square*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "\mathsf{inr}"] \\ A \arrow[r, "\mathsf{inl}"'] & A \sqcup_C B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

*If $f$ belongs to $\mathcal{L}$, then so does* $\mathsf{inr}$.

*Proof.* We must prove that for each lifting problem

$$\begin{tikzcd} B \arrow[r, "t"] \arrow[d, "\mathsf{inr}"'] \arrow[dr, phantom, "S"] & E \arrow[d, "v"] \\ A \sqcup_C B \arrow[r, "b"'] & H \end{tikzcd}$$

with $v \in \mathcal{R}$, the type $\mathsf{fill}(S)$ is contractible. Note that the type

$$\Phi \ := \ \mathsf{fill}(\lambda x.\, \mathsf{ap}_b(\mathsf{glue}(x)) \cdot S(g(x)))$$

of fillers for the composite diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "t"] & E \arrow[d, "v"] \\ A \arrow[r, "\mathsf{inl}"'] & A \sqcup_C B \arrow[r, "b"'] & H \end{tikzcd}$$

is contractible because $f \in \mathcal{L}$. By the induction principle for pushouts, we find that $\mathsf{fill}(S)$ is equivalent to the type of data

$$\begin{aligned} k \ &: \ A \to E \\ K_1 \ &: \ k \circ f \sim t \circ g \\ K_2 \ &: \ v \circ k \sim b \circ \mathsf{inl} \\ K_3 \ &: \ \prod_{x:C} \mathsf{ap}_v(K_1(x)) = K_2(f(x)) \cdot \mathsf{ap}_b(\mathsf{glue}(x)) \cdot S(g(x)) \end{aligned}$$

This type is exactly $\Phi$, and thus $\mathsf{fill}(S)$ is contractible. $\square$

Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be 0-functors of wild categories. An adunction $L \dashv R$ consists of terms

$$\begin{aligned}
\alpha &: \prod_{A:\mathsf{Ob}(\mathcal{C})} \prod_{X:\mathsf{Ob}(\mathcal{D})} \mathsf{hom}_{\mathcal{D}}(LA, X) \simeq \mathsf{hom}_{\mathcal{C}}(A, RX) \\
V_1 &: \prod_{A:\mathsf{Ob}(\mathcal{C})} \prod_{X,Y:\mathsf{Ob}(\mathcal{D})} \prod_{g:\mathsf{hom}_{\mathcal{D}}(X,Y)} \prod_{h:\mathsf{hom}_{\mathcal{D}}(LA,X)} Rg \circ \alpha(h) = \alpha(g \circ h) \\
V_2 &: \prod_{Y:\mathsf{Ob}(\mathcal{D})} \prod_{A,B:\mathsf{Ob}(\mathcal{C})} \prod_{f:\mathsf{hom}_{\mathcal{C}}(A,B)} \prod_{h:\mathsf{hom}_{\mathcal{D}}(LB,Y)} \alpha(h) \circ f = \alpha(h \circ Lf).
\end{aligned}$$

Note that for each such triple, we also have naturality squares

$$\begin{tikzcd}
\mathsf{hom}_{\mathcal{C}}(A, RX) \arrow[r, "Rg\circ -"] \arrow[d, "\alpha^{-1}"'] \arrow[dr, phantom, "\widetilde{V}_1(g)"] & \mathsf{hom}_{\mathcal{C}}(A, RY) \arrow[d, "\alpha^{-1}"] \\
\mathsf{hom}_{\mathcal{D}}(LA, X) \arrow[r, "g\circ -"'] & \mathsf{hom}_{\mathcal{D}}(LA, Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{hom}_{\mathcal{C}}(B, RY) \arrow[r, "-\circ f"] \arrow[d, "\alpha^{-1}"'] \arrow[dr, phantom, "\widetilde{V}_2(f)"] & \mathsf{hom}_{\mathcal{C}}(A, RY) \arrow[d, "\alpha^{-1}"] \\
\mathsf{hom}_{\mathcal{D}}(LB, Y) \arrow[r, "-\circ Lf"'] & \mathsf{hom}_{\mathcal{D}}(LA, Y)
\end{tikzcd}$$

Here, we have defined

$$\begin{aligned}
\widetilde{V}_1(g,h) &:= \eta(g \circ \alpha^{-1}(h))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_1(g, \alpha^{-1}(h)))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{Rg\circ -}(\epsilon(h))) \\
\widetilde{V}_2(f,h) &:= \eta(\alpha^{-1}(h) \circ Lf)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_2(f, \alpha^{-1}(h)))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon(h)))
\end{aligned}$$

using the half-adjoint equivalence data $\eta$ and $\epsilon$ enjoyed by $\alpha$.

Suppose that $(\alpha, V_1, V_2) : L \dashv R$. Let $f : \mathsf{hom}_{\mathcal{C}}(A, B)$ and $g : \mathsf{hom}_{\mathcal{D}}(X, Y)$. Define

$$\begin{aligned}
\varphi &: \sum_{u:\mathsf{hom}_{\mathcal{D}}(LA,X)} \sum_{v:\mathsf{hom}_{\mathcal{D}}(LB,Y)} v \circ Lf = g \circ u \;\to\; \sum_{r:\mathsf{hom}_{\mathcal{C}}(A,RX)} \sum_{s:\mathsf{hom}_{\mathcal{C}}(B,RY)} s \circ f = Rg \circ r \\
\varphi(u, v, G) &:= \big(\alpha(u), \alpha(v), V_2(f, v) \cdot \mathsf{ap}_{\alpha}(G) \cdot V_1(g, u)^{-1}\big).
\end{aligned}$$

**Lemma 3.2.8.** *The function $\varphi$ is an equivalence, so that*

$$\mathsf{fill}(G) \simeq \mathsf{fill}(V_2(f, v) \cdot \mathsf{ap}_{\alpha}(G) \cdot V_1(g, u)^{-1})$$

*for each $G : v \circ Lf = g \circ u$.*

*Proof.* Define

$$\psi \ : \sum_{u:\mathsf{hom}_{\mathcal{C}}(A,RX)} \sum_{v:\mathsf{hom}_{\mathcal{C}}(B,RY)} v \circ f = Rg \circ u \ \to \sum_{r:\mathsf{hom}_{\mathcal{D}}(LA,X)} \sum_{s:\mathsf{hom}_{\mathcal{D}}(LB,Y)} s \circ Lf = g \circ r$$

$$\psi(u,v,G) \ :\equiv \ \left(\alpha^{-1}(u), \alpha^{-1}(v), \widetilde{V}_2(f,v) \cdot \mathsf{ap}_{\alpha^{-1}}(G) \cdot \widetilde{V}_1(g,u)^{-1}\right).$$

We claim that $\psi$ is a quasi-inverse (i.e., two-sided inverse) of $\varphi$.

We have that

$$\begin{gathered}
\mathsf{ap}_{\alpha^{-1}}(V_2(f,v) \cdot \mathsf{ap}_{\alpha}(G) \cdot V_1(g,u)^{-1}) \\
\| \\
\mathsf{ap}_{\alpha^{-1}}(V_2(f,v)) \cdot \mathsf{ap}_{\alpha^{-1}\circ\alpha}(G) \cdot \mathsf{ap}_{\alpha^{-1}}(V_1(g,u))^{-1} \\
\| \\
\mathsf{ap}_{\alpha^{-1}}(V_2(f,v)) \cdot \eta(v \circ Lf) \cdot G \cdot \eta(g \circ u)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_1(g,u))^{-1}
\end{gathered}$$

$$\begin{gathered}
\widetilde{V}_2(f, \alpha(v)) \\
\|\text{definitional} \\
\eta(\alpha^{-1}(\alpha(v)) \circ Lf)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_2(f, \alpha^{-1}(\alpha(v))))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon(\alpha(v)))) \\
\| \\
\eta(\alpha^{-1}(\alpha(v)) \circ Lf)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\mathsf{ap}_{\alpha}(\eta(v))) \cdot V_2(f,v) \cdot \mathsf{ap}_{\alpha}(\mathsf{ap}_{-\circ Lf}(\eta(v)))^{-1})^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon(\alpha(v)))) \\
\| \\
\eta(\alpha^{-1}(\alpha(v)) \circ Lf)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}\circ\alpha}(\mathsf{ap}_{-\circ Lf}(\eta(v))) \cdot \mathsf{ap}_{\alpha^{-1}}(V_2(f,v))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\mathsf{ap}_{\alpha}(\eta(v))))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon(\alpha(v)))) \\
\| \\
\mathsf{ap}_{-\circ Lf}(\eta(v)) \cdot \eta(v \circ Lf)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_2(f,v))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\mathsf{ap}_{\alpha}(\eta(v))))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon(\alpha(v)))) \\
\| \\
\mathsf{ap}_{-\circ Lf}(\eta(v)) \cdot \eta(v \circ Lf)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_2(f,v))^{-1}
\end{gathered}$$

$$\begin{gathered}
\widetilde{V}_1(g, \alpha(u)) \\
\|\text{definitional} \\
\eta(g \circ \alpha^{-1}(\alpha(u)))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_1(g, \alpha^{-1}(\alpha(u))))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{Rg\circ -}(\epsilon(\alpha(u)))) \\
\| \\
\eta(g \circ \alpha^{-1}(\alpha(u)))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{Rg\circ -}(\mathsf{ap}_{\alpha}(\eta(u))) \cdot V_1(g,u) \cdot \mathsf{ap}_{\alpha}(\mathsf{ap}_{g\circ -}(\eta(u)))^{-1})^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{Rg\circ -}(\epsilon(\alpha(u)))) \\
\| \\
\mathsf{ap}_{g\circ -}(\eta(u)) \cdot \eta(g \circ u)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_1(g,u))^{-1}
\end{gathered}$$

which implies that

$$
\begin{gathered}
\widetilde{V}_2(f,\alpha(v)) \cdot \mathsf{ap}_{\alpha^{-1}}(V_2(f,v) \cdot \mathsf{ap}_\alpha(G) \cdot V_1(g,u)^{-1}) \cdot \widetilde{V}_1(g,\alpha(u))^{-1} \\
\| \\
\widetilde{V}_2(f,\alpha(v)) \cdot \big(\mathsf{ap}_{\alpha^{-1}}(V_2(f,v)) \cdot \eta(v \circ Lf) \cdot G \cdot \eta(g \circ u)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_1(g,u))^{-1}\big) \cdot \widetilde{V}_1(g,\alpha(u))^{-1} \\
\| \\
\big(\mathsf{ap}_{-\circ Lf}(\eta(v)) \cdot \eta(v \circ Lf)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_2(f,v))^{-1}\big) \cdot \big(\mathsf{ap}_{\alpha^{-1}}(V_2(f,v)) \cdot \eta(v \circ Lf) \cdot G \cdot \eta(g \circ u)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_1(g,u))^{-1}\big) \cdot \big(\mathsf{ap}_{g\circ -}(\eta(u)) \cdot \eta(g \circ u)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_1(g,u))^{-1}\big)^{-1} \\
\| \\
\mathsf{ap}_{-\circ Lf}(\eta(v)) \cdot G \cdot \mathsf{ap}_{g\circ -}(\eta(u))^{-1}
\end{gathered}
$$

This proves that $\psi \circ \varphi \sim \mathsf{id}$. Similarly, $\varphi \circ \psi \sim \mathsf{id}$. $\square$

**Corollary 3.2.9.** *Suppose that both $\mathcal{C}$ and $\mathcal{D}$ are univalent bicategories with OFS's $(\mathcal{L}_1, \mathcal{R}_1)$ and $(\mathcal{L}_2, \mathcal{R}_2)$, respectively. The 0-functor $R$ preserves $\mathcal{R}$ if and only if $L$ preserves $\mathcal{L}$.*

*Proof.* Suppose that $R$ preserves $\mathcal{R}$. Let $f : \mathsf{hom}_{\mathcal{C}}(A,B)$ such that $f \in \mathcal{L}_1$. Consider a commuting square

$$
\begin{tikzcd}
LA \arrow[r, "u"] \arrow[d, "Lf"'] \arrow[dr, phantom, "S"] & X \arrow[d, "g"] \\
LB \arrow[r, "v"'] & Y
\end{tikzcd}
$$

where $g \in \mathcal{R}_2$. By Corollary 3.2.6, if $\mathsf{fill}(S)$ is contractible, then $Lf \in \mathcal{L}_2$. By Lemma 3.2.8, this type is equivalent to the type of fillers of the square

$$
\begin{tikzcd}
A \arrow[r, "\alpha(u)"] \arrow[d, "f"'] \arrow[dr, phantom, "V_2(f,v)\cdot\mathsf{ap}_\alpha(G)\cdot V_1(g,u)^{-1}"] & RX \arrow[d, "Rg"] \\
B \arrow[r, "\alpha(v)"'] & RY
\end{tikzcd}
$$

By Corollary 3.2.6 again, this is contractible because $Rg \in \mathcal{R}_1$.

The converse is formally dual. $\square$

## 3.3 Coslices of $\mathcal{U}$

Let $\mathcal{U}$ be a universe and let $A : \mathcal{U}$. Suppose that $X$ and $Y$ are elements of $A/\mathcal{U} :\equiv \sum_{X:\mathcal{U}} (A \to X)$. Consider the type

$$X \to_A Y \;:\equiv\; \sum_{h:\mathsf{pr}_1(X)\to\mathsf{pr}_1(Y)} h \circ \mathsf{pr}_2(X) \sim \mathsf{pr}_2(Y)$$

of maps from $X$ to $Y$. In particular, note that

$$X \to_{\mathbf{1}} Y \quad = \quad (\mathsf{pr}_1(X), \mathsf{pr}_2(X)(*)) \to_* (\mathsf{pr}_1(Y), \mathsf{pr}_2(Y)(*))$$

the type of pointed maps from $X$ to $Y$.

Suppose that $g : X \to_A Y$ and $h : Y \to_A Z$. The *composite of $g$ and $h$* is the term

$$h \circ g :\equiv \Big(\mathsf{pr}_1(h) \circ \mathsf{pr}_1(g), \lambda a.\, \mathsf{ap}_{\mathsf{pr}_1(h)}(\mathsf{pr}_2(g)(a)) \cdot \mathsf{pr}_2(h)(a)\Big) \;:\; X \to_A Z$$

This gives us a bicategory $A/\mathcal{U}$, called the *coslice of $\mathcal{U}$ under $A$*.

**Example 3.3.1.**

- The coslice $\mathbf{1}/\mathcal{U}$ is known as the category of *pointed types*. We may denote it by $\mathcal{U}^*$.
- The coslice $\mathbf{2}/\mathcal{U}$ is known as the category of *bipointed types*.

**Proposition 3.3.2.** *For all $X, Y : A/\mathcal{U}$, we have an equivalence*

$$(X = Y) \;\simeq\; \sum_{k : \mathsf{pr}_1(X) \xrightarrow{\simeq} \mathsf{pr}_1(Y)} k \circ \mathsf{pr}_2(X) \sim \mathsf{pr}_2(Y)$$

**Definition 3.3.3.** Let $f, g : X \to_A Y$. An *$A$-homotopy* $f \sim_A g$ between $f$ and $g$ is an ordinary homotopy $H : \mathsf{pr}_1(f) \sim \mathsf{pr}_1(g)$ together with a path

$$H(\mathsf{pr}_2(X)(a))^{-1} \cdot \mathsf{pr}_2(f)(a) \;=\; \mathsf{pr}_2(g)(a)$$

for each $a : A$.

**Lemma 3.3.4.** *For all $f, g : X \to_A Y$, the function*

$$\mathsf{happly}_A \;:\; (f = g) \to (f \sim_A g)$$

*defined by path induction is an equivalence.*

*Proof.* For each $k : \mathsf{pr}_1(X) \to \mathsf{pr}_1(Y)$, the total space

$$\sum_{g : \mathsf{pr}_1(X) \to \mathsf{pr}_1(Y)} k \sim g$$

is contractible because $\mathsf{happly}$ is an equivalence by function extensionality. Thus, by Theorem A.0.3, we just need to observe that

$$\sum_{g_p : \prod_{a:A} \mathsf{pr}_1(f) \circ \mathsf{pr}_2(X) = \mathsf{pr}_2(Y)} \mathsf{pr}_2(f) \sim g_p$$

is contractible for all $f : X \to_A Y$. □

*Notation.* Define $\langle H, p \rangle :\equiv \mathsf{happly}_A^{-1}(H, p)$.

## 3.4 Diagrams in coslices

Let $\Gamma$ be a graph. An $A$-*diagram over* $\Gamma$ consists of a family $F : \Gamma_0 \to A/\mathcal{U}$ of objects in $A/\mathcal{U}$ and a map $F_{i,j,g} : F_i \to_A F_j$ for all $i, j : \Gamma_0$ and $g : \Gamma_1(i, j)$.

Let $F$ be an $A$-diagram over $\Gamma$ and let $C : A/\mathcal{U}$. A *cocone under* $F$ *on* $C$ consists of a family of maps $r : \prod_{i:\Gamma_0} F_i \to_A C$ equipped with

- for each $i, j : \Gamma_0$ and $g : \Gamma_1(i, j)$, a homotopy $h_{i,j,g} : \mathsf{pr}_1(r_j) \circ \mathsf{pr}_1(F_{i,j,g}) \sim \mathsf{pr}_1(r_i)$
- for each $a : A$, a path

$$h_{i,j,g}(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a) \;=\; \mathsf{ap}_{\mathsf{pr}_1(r_i)}(a)$$

Let $\mathsf{Cocone}_F(C)$ denote the type of cocones under $F$ on $C$.

**Lemma 3.4.1.** *For all* $(\alpha, p), (\beta, q) : \mathsf{Cocone}_F(C)$*, we have an an equivalence between the identity type* $(\alpha, p) =_{\mathsf{Cocone}_F(C)} (\beta, q)$ *and the type of data*

$$\begin{aligned}
W \;&:\; \prod_{i:\Gamma_0} \mathsf{pr}_1(\alpha_i) \sim \mathsf{pr}_1(\beta_i) \\
u \;&:\; \prod_{i:\Gamma_0} \prod_{a:A} W_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{pr}_2(\alpha_i)(a) = \mathsf{pr}_2(\beta_i)(a)
\end{aligned}$$

*for all* $i, j : \Gamma_0,\ g : \Gamma_1(i, j)\ \ldots$

$$\begin{aligned}
S_1(i,j,g) \;&:\; \prod_{x:\mathsf{pr}_1(F_i)} W_j(\mathsf{pr}_1(F_{i,j,g})(x))^{-1} \cdot \mathsf{pr}_1(p_{i,j,g})(x) \cdot W_i(x) = \mathsf{pr}_1(q_{i,j,g})(x) \\
S_2(i,j,g) \;&:\; \prod_{a:A} \mathsf{ap}_{_^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\beta_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\beta_j)(a)}(S_1(i,j,g,\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \Xi(W,u,p_{i,j,g},a) = \mathsf{pr}_2(q_{i,j,g})(a)
\end{aligned}$$

*Here,* $\Xi(W, u, p_{i,j,g}, a)$ *denotes the chain of equalities*

$$\begin{gathered}
\Big(W_j(\mathsf{pr}_1(F_{i,j,g})(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)) \cdot W_i(\mathsf{pr}_2(F_i)(a))\Big)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\beta_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\beta_j)(a) \\
\Big\Vert \\
{\scriptstyle \mathsf{Pl}(\mathsf{pr}_2(F_{i,j,g})(a),\, W_j(\mathsf{pr}_2(F_j)(a)))} \\
\Big\Vert \\
\Big(\Big(\mathsf{ap}_{\mathsf{pr}_1(\beta_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot W_j(\mathsf{pr}_2(F_j)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\Big) \cdot \mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)) \cdot W_i(\mathsf{pr}_2(F_i)(a))\Big)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\beta_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\beta_j)(a) \\
\Big\Vert \\
{\scriptstyle \textit{via } u_j(a)} \\
\Big\Vert \\
\Big(\Big(\mathsf{ap}_{\mathsf{pr}_1(\beta_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \big(\mathsf{pr}_2(\beta_j)(a) \cdot \mathsf{pr}_2(\alpha_j)(a)^{-1}\big) \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\Big) \cdot \mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)) \cdot W_i(\mathsf{pr}_2(F_i)(a))\Big)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\beta_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\beta_j)(a) \\
\Big\Vert \\
{\scriptstyle \mathsf{Pl}(\mathsf{pr}_2(F_{i,j,g})(a),\, \mathsf{pr}_2(\beta_j)(a),\, \mathsf{pr}_2(\alpha_j)(a),\, \mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)),\, W_i(\mathsf{pr}_2(F_i)(a)))} \\
\Big\Vert \\
W_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\alpha_j)(a) \\
\Big\Vert \\
{\scriptstyle \mathsf{ap}_{W_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot _}(\mathsf{pr}_2(p_{i,j,g})(a))} \\
\Big\Vert \\
W_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{pr}_2(\alpha_i)(a) \\
\Big\Vert \\
{\scriptstyle u_i(a)} \\
\Big\Vert \\
\mathsf{pr}_2(\beta_i)(a)
\end{gathered}$$

*Proof.* First, we have an equivalence

$$(\alpha, p) = (\beta, q) \quad \simeq \quad \sum_{H : \prod_{i:\Gamma_0} \alpha_i \sim_A \beta_i} \prod_{i,j,g} \mathsf{transp}^{K \mapsto F_{i,j,g} \circ K_j \sim_A K_i}(\mathsf{funext}(\lambda i.\, \mathsf{happly}_A^{-1}(H_i)), p_{i,j,g}) = q_{i,j,g}$$

by Theorem A.0.3. Indeed, the type family

$$\beta \;\mapsto\; \prod_{i:\Gamma_0} \alpha_i \sim_A \beta_i$$

pointed by $i \mapsto \big(\mathsf{refl}_{\mathsf{pr}_1(\alpha_i)(x)}, \mathsf{refl}_{\mathsf{pr}_2(\alpha_i)(a)}\big)$ is an identity system on $\prod_{i:\Gamma_0} \mathsf{pr}_1(F_i) \to_A C$ as

$$\prod_{i:\Gamma_0} \alpha_i \sim_A \beta_i \;\simeq\; \prod_{i:\Gamma_0} \alpha_i = \beta_i \;\simeq\; \alpha = \beta$$

by Lemma 3.3.4. Further, it's easy to check that the type family

$$(\beta, q, H) \;\mapsto\; \prod_{i,j,g} \mathsf{transp}^{K \mapsto F_{i,j,g} \circ K_j \sim_A K_i}(\mathsf{funext}(\lambda i.\, \mathsf{happly}_A^{-1}(H_i)), p_{i,j,g}) = q_{i,j,g}$$

is an SNS on

$$\left(\beta \mapsto \prod_{i,j,g} F_{i,j,g} \circ K_j \sim_A K_i, \beta \mapsto \prod_{i:\Gamma_0} \alpha_i \sim_A \beta_i\right)$$

Thus, it suffices to observe that

$$\sum_{q : \prod_{i,j,g} F_{i,j,g} \circ K_j \sim_A K_i} \prod_{i,j,g} p_{i,j,g} = q_{i,j,g}$$

is contractible.

Next, note that

$$\textstyle\sum_{H : \prod_{i:\Gamma_0} \alpha_i \sim_A \beta_i} \prod_{i,j,g} \mathsf{transp}^{K \mapsto F_{i,j,g} \circ K_j \sim_A K_i}(\mathsf{funext}(\lambda i.\, \mathsf{happly}_A^{-1}(H_i)), p_{i,j,g}) = q_{i,j,g}$$

$$\wr|$$

$$\textstyle\sum_{W : \prod_{i:\Gamma_0} \mathsf{pr}_1(\alpha_i) \sim \mathsf{pr}_1(\beta_i)} \sum_{u : \prod_{i:\Gamma_0} \prod_{a:A} W_j(\mathsf{pr}_1(F_{i,j,g})(x))^{-1} \cdot \mathsf{pr}_1(p_{i,j,g})(x) \cdot W_i(x) = \mathsf{pr}_1(q_{i,j,g})(x)} \prod_{i,j,g} \mathsf{transp}^{K \mapsto F_{i,j,g} \circ K_j \sim_A K_i}(\mathsf{funext}(\lambda i.\, \langle W_i, u_i \rangle), p_{i,j,g}) = q_{i,j,g}$$

Thus, it suffices to find an equivalence

$$\mathsf{transp}^{K \mapsto F_{i,j,g} \circ K_j \sim_A K_i}(\mathsf{funext}(\lambda i.\, \langle W_i, u_i \rangle), p_{i,j,g}) = q_{i,j,g}$$

$$\wr|$$

$$\textstyle\sum_{S : \prod_{x:\mathsf{pr}_1(F_i)} W_j(\mathsf{pr}_1(F_{i,j,g})(x))^{-1} \cdot \mathsf{pr}_1(p_{i,j,g})(x) \cdot W_i(x) = \mathsf{pr}_1(q_{i,j,g})(x)} \prod_{a:A} \mathsf{ap}_{\_^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\beta_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\beta_j)(a)}(S(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \Xi(W, u, p_{i,j,g}, a) = \mathsf{pr}_2(q_{i,j,g})(a)$$

for all $W : \prod_{i:\Gamma_0} \mathsf{pr}_1(\alpha_i) \sim \mathsf{pr}_1(\beta_i)$, $u : \prod_{i:\Gamma_0} \prod_{a:A} W_j(\mathsf{pr}_1(F_{i,j,g})(x))^{-1} \cdot \mathsf{pr}_1(p_{i,j,g})(x) \cdot W_i(x) = \mathsf{pr}_1(q_{i,j,g})(x)$, $i, j : \Gamma_0$, and $g : \Gamma_1(i,j)$. We can do this by generalizing the data of the desired equivalence to a form suitable for induction on the identity system

$$B \mapsto \prod_{i:\Gamma_0} \alpha_i \sim_A B_i$$

on $\left(\prod_{i:\Gamma_0} F_i \to_A C, \alpha\right)$. We claim that for all $B : \prod_{i:\Gamma_0} F_i \to_A C$, $e : \prod_{i:\Gamma_0} \alpha_i \sim_A B_i$, and $r : F_{i,j,g} \circ K_j \sim_A K_i$, we have an equivalence

$$\begin{gathered}\mathsf{transp}^{K \mapsto F_{i,j,g} \circ K_j \sim_A K_i}(\mathsf{funext}(\lambda i.\, \mathsf{funext}_A(e_i)), p_{i,j,g}) = r \\ \wr\! \mid \\ \textstyle\sum_{S:\prod_{x:\mathsf{pr}_1(F_i)} \mathsf{pr}_1(e_j)(\mathsf{pr}_1(F_{i,j,g})(x))^{-1} \cdot \mathsf{pr}_1(p_{i,j,g})(x) \cdot \mathsf{pr}_1(e_i)(x) = \mathsf{pr}_1(r)(x)} \prod_{a:A} \mathsf{ap}_{-^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(B_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(B_j)(a)}(S(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \Xi(\lambda i.\, \mathsf{pr}_1(e_i), \lambda i.\, \mathsf{pr}_2(e_i), p_{i,j,g}, a) = \mathsf{pr}_2(r)(a)\end{gathered}$$

Now, by path induction on $e$, we want to find an equivalence

$$\begin{gathered}p_{i,j,g} =_{F_{i,j,g} \circ K_j \sim_A K_i} r \\ \wr\! \mid \\ \textstyle\sum_{S:\prod_{x:\mathsf{pr}_1(F_i)} \mathsf{pr}_1(p_{i,j,g})(x) \cdot \mathsf{refl}_{\alpha_i(x)} = \mathsf{pr}_1(r)(x)} \prod_{a:A} \mathsf{ap}_{-^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\alpha_j)(a)}(S(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \Xi(\lambda i \lambda x.\mathsf{refl}_{\mathsf{pr}_1(\alpha_i)(x)}, \lambda i \lambda a.\mathsf{refl}_{\mathsf{pr}_2(\alpha_i)(a)}, p_{i,j,g}, a) = \mathsf{pr}_2(r)(a)\end{gathered}$$

We can use Theorem A.0.3 to find an equivalence

$$\begin{gathered}p_{i,j,g} = r \\ \wr\! \mid \\ \textstyle\sum_{S:\prod_{x:\mathsf{pr}_1(F_i)} \mathsf{pr}_1(p_{i,j,g})(x) \cdot \mathsf{refl}_{\alpha_i(x)} = \mathsf{pr}_1(r)(x)} \prod_{a:A} \mathsf{ap}_{-^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\alpha_j)(a)}(S(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{RId}(\mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)))) \cdot \mathsf{pr}_2(p_{i,j,g})(a) = \mathsf{pr}_2(r)(a)\end{gathered}$$

Thus, it suffices to prove that

$$\begin{gathered}\mathsf{pr}_2(p_{i,j,g})(a) \\ \Vert \\ \mathsf{ap}_{-^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\alpha_j)(a)}(\mathsf{RId}(\mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \Xi(\lambda i \lambda x.\mathsf{refl}_{\mathsf{pr}_1(\alpha_i)(x)}, \lambda i \lambda a.\mathsf{refl}_{\mathsf{pr}_2(\alpha_i)(a)}, p_{i,j,g}, a) = \mathsf{pr}_2(r)(a)\end{gathered}$$

for all $S$ and $a$. By unfolding $\Xi$, this identity becomes

$$\begin{tikzcd}
\mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\alpha_j)(a) \arrow[d, equal, "\mathsf{ap}_{_^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\alpha_j)(a)}(\mathsf{RId}(\mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a))))^{-1}"] \arrow[dddddd, bend right, dotted, "\mathsf{pr}_2(p_{i,j,g})(a)"'] \\
\left(\mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)) \cdot \mathsf{refl}_{\mathsf{pr}_1(\alpha_i)(\mathsf{pr}_2(F_i)(a))}\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\alpha_j)(a) \arrow[d, equal, "\mathsf{Pl}(\mathsf{pr}_2(F_{i,j,g})(a))"] \\
\left(\left(\mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\right) \cdot \mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)) \cdot \mathsf{refl}_{\mathsf{pr}_1(\alpha_i)(\mathsf{pr}_2(F_i)(a))}\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\alpha_j)(a) \arrow[d, equal, "\mathsf{Pl}(\mathsf{pr}_2(\alpha_j)(a))"] \\
\left(\left(\mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot (\mathsf{pr}_2(\alpha_j)(a) \cdot \mathsf{pr}_2(\alpha_j)(a)^{-1}) \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\right) \cdot \mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)) \cdot \mathsf{refl}_{\mathsf{pr}_1(\alpha_i)(\mathsf{pr}_2(F_i)(a))}\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\alpha_j)(a) \arrow[d, equal, "\mathsf{Pl}(\mathsf{pr}_2(F_{i,j,g})(a), \mathsf{pr}_2(\alpha_j)(a), \mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)))"] \\
\mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\alpha_j)(a) \arrow[d, equal, "\mathsf{pr}_2(p_{i,j,g})(a)"] \\
\mathsf{pr}_2(\alpha_i)(a)
\end{tikzcd}$$

We want to prove that composing the first four paths of this chain returns the trivial path. To see this, do iterative path induction on the data

$$\begin{aligned}
\mathsf{pr}_2(F_{i,j,g})(a) &\ :\ \mathsf{pr}_1(F_{i,j,g})(\mathsf{pr}_2(F_i)(a)) =_{\mathsf{pr}_1(F_j)} \Phi_1 \\
\mathsf{pr}_2(\alpha_j)(a) &\ :\ \mathsf{pr}_1(\alpha_j)(\Phi_1) =_{\mathsf{pr}_1(C)} \Phi_2 \\
\mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)) &\ :\ \mathsf{pr}_1(\alpha_j)(\mathsf{pr}_1(F_{i,j,g})(\mathsf{pr}_2(F_i)(a))) =_{\mathsf{pr}_1(C)} \Phi_3
\end{aligned}$$

with free endpoints marked by $\Phi$. This reduces our goal to

$$\mathsf{refl}_{\mathsf{refl}_{\mathsf{pr}_1(\alpha_j)(\mathsf{pr}_1(F_{i,j,g})(\mathsf{pr}_2(F_i)(a)))}} = \mathsf{refl}_{\mathsf{refl}_{\mathsf{pr}_1(\alpha_j)(\mathsf{pr}_1(F_{i,j,g})(\mathsf{pr}_2(F_i)(a)))}}$$

which completes the proof. $\square$

**Note 3.4.2.**

(a) Let $F$ and $G$ be $A$-diagrams over a graph $\Gamma$. The type of *natural transformations* from $F$ to $G$ is

$$F \Rightarrow_A G \ \ :\equiv \sum_{\alpha : \prod_{i:\Gamma_0} \mathsf{pr}_1(F_i) \to_A \mathsf{pr}_1(G_i)} \ \prod_{i,j:\Gamma_0} \ \prod_{g:\Gamma_1(i,j)} (G_{i,j,g} \circ \alpha_i \sim_A \alpha_j \circ F_{i,j,g})$$

For each $C : A/\mathcal{U}$, we have an evident equivalence $\mathsf{Cocone}_F(C) \ \simeq \ (F \Rightarrow_A \mathsf{const}_\Gamma(C))$.

(b) For every graph $\Gamma$ and every $\mathcal{U}$-valued diagram $F$ over $\Gamma$, recall the (standard) limit of $F$ [2, Definition 4.2.7],

$$\mathsf{lim}_\Gamma(F) \ :\equiv \sum_{x : \prod_{i:\Gamma_0} F_i} \ \prod_{i,j:\Gamma_0} \ \prod_{g:\Gamma_1(i,j)} F_{i,j,g}(x_i) =_{F_j} x_j$$

which is functorial in $F$.

Let $F$ be an $A$-diagram over a graph $\Gamma$ and let $C : A/\mathcal{U}$. We have another evident equivalence

$$\mathsf{Cocone}_F(C) \;\simeq\; \mathsf{lim}_{i:\Gamma^{\mathrm{op}}}(F_i \to_A C)$$

# 4 Colimits

Colimits inside HoTT are interpreted as (internal) colimits over free categories on quivers.

## 4.1 Colimits in $\mathcal{U}$

The colimit $\mathsf{colim}_\Gamma(F)$ of a diagram $F$ in $\mathcal{U}$ over $\Gamma$ is the HIT generated by

$$\begin{aligned} \iota &: (i : \Gamma_0) \to F_i \to \mathsf{colim}_\Gamma(F) \\ \kappa &: (i, j : \Gamma_0)\,(g : \Gamma_1(i,j)) \to \iota_j \circ F_{i,j,g} \sim \iota_i \end{aligned}$$

$$\begin{tikzcd} F_i \arrow[rr, "F_{i,j,g}"] \arrow[dr, "\iota_i"'] \arrow[rr, phantom, shift right=6, "\kappa_{i,j,g}"] & & F_j \arrow[dl, "\iota_j"] \\ & \mathsf{colim}_\Gamma(F) & \end{tikzcd}$$

We have the following induction principle.

$$\begin{aligned} E &: \mathsf{colim}_\Gamma(F) \to \mathcal{U} \\ e &: \prod_{i:\Gamma_0} \prod_{x:F_i} E(\iota_i(x)) \\ q &: \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} \prod_{x:F_i} \mathsf{transp}^E(\kappa_{i,j,g}(x), e_j(F_{i,j,g}(x))) = e_i(x) \\ & \Downarrow \\ \mathsf{ind}(E,e,q) &: \prod_{z:\mathsf{colim}_\Gamma(F)} E(z), \quad \mathsf{ind}(E,e,q)(\iota_i(x)) \equiv e_i(x),\; i:\Gamma_0,\; x:F_i \\ \rho_e(q) &: \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} \prod_{x:F_i} \mathsf{apd}_{\mathsf{ind}(E,e,q)}(\kappa_{i,j,g}(x)) = q_{i,j,g}(x) \end{aligned}$$

We also have the following recursion principle.

$$\begin{aligned}
E &: \mathcal{U} \\
e &: \prod_{i:\Gamma_0} F_i \to E \\
q &: \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} \prod_{x:F_i} e_j(F_{i,j,g}(a)) = e_i(a) \\
&\Downarrow \\
\mathsf{rec}(E,e,q) &: \mathsf{colim}_\Gamma(F) \to E, \quad \mathsf{rec}(E,e,q)(\iota_i(x)) \equiv e_i(x),\ i:\Gamma_0,\ x:F_i \\
\rho_e(q) &: \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} \prod_{x:F_i} \mathsf{ap}_{\mathsf{rec}(E,e,q)}(\kappa_{i,j,g}(x)) = q_{i,j,g}(x).
\end{aligned}$$

**Example 4.1.1.**

1. If $\Gamma_0 \equiv \mathbb{N}$ and $\Gamma_1(i,j) \equiv\ i+1=j$, then $\Gamma$ is precisely the ordinal $\omega$. (We may abuse notation by referring to $\omega$ as just $\mathbb{N}$.)

   For any type family $F : \mathbb{N} \to \mathcal{U}$, we have an equivalence

$$\epsilon : \underbrace{\left(\prod_{n,m:\mathbb{N}} (n+1=m) \to F_n \to F_m\right)}_{\text{diagrams over } \omega} \xrightarrow{\simeq} \left(\prod_{n:\mathbb{N}} F_n \to F_{n+1}\right)$$

$$\epsilon(F) \ :\equiv\ n \mapsto F_{n,n+1}(\mathsf{refl}_{n+1})$$

   along with an equivalence $\mathsf{colim}(F) \simeq \overbrace{\mathsf{colim}_{\mathsf{seq}}(\epsilon(F))}^{\text{sequential colimit}}$ for every diagram $F$ over $\omega$. Specifically, construct a quasi-inverse of $\epsilon$ by sending each $f : \prod_{n:\mathbb{N}} F_n \to F_{n+1}$ to

$$\lambda n \lambda m \lambda g.\, \mathsf{transp}^{k \mapsto F_n \to F_k}(g, f_n) : \prod_{n,m:\mathbb{N}} (n+1=m) \to F_n \to F_m$$

2. If $\Gamma_0 \equiv \{l, r, m\}$ (i.e., $\mathsf{Fin}(3)$) and

$$\begin{aligned}
\Gamma_1(m,l) &\equiv \mathbf{1} \\
\Gamma_1(m,r) &\equiv \mathbf{1} \\
\Gamma_1(i,j) &\equiv \mathbf{0} \quad \text{otherwise}
\end{aligned}$$

   then $\mathsf{colim}(F)$ is equivalent to the pushout $F(l) \sqcup_{F(m)} F(r)$, i.e., the HIT generated by the functions

   - $\mathsf{left} : F(l) \to F(l) \sqcup_{F(m)} F(r)$
   - $\mathsf{right} : F(r) \to F(l) \sqcup_{F(m)} F(r)$
   - $\mathsf{glue} : \prod_{x:F(m)} (\mathsf{left}(F_{m,l}(*)(x)) = \mathsf{right}(F_{m,r}(*)(x)))$.

3. If $\Gamma_0$ is a type and $\Gamma_1(i,j) \equiv \mathbf{0}$ for all $i, j : \Gamma_0$, then $\Gamma$ is called the *discrete graph on* $\Gamma_0$. In this case, $\mathsf{colim}_\Gamma(F)$ is equivalent to the coproduct $\sum_{i:\Gamma_0} F_i$.

**Lemma 4.1.2.** *Let $\Gamma$ be a graph. Suppose that $F$ is a diagram over $\Gamma$. Let $Z$ be a type and $h_1, h_2 : \mathsf{colim}_\Gamma(F) \to Z$. If we have a term*

$$p_i(x) \;:\; h_1(\iota_i(x)) = h_2(\iota_i(x))$$

*for all $i : \Gamma_0$ and $x : F_i$ along with a commuting square*

$$\begin{tikzcd}
h_1(\iota_j(F_{i,j,g}(x))) \arrow[r, equal, "\mathsf{ap}_{h_1}(\kappa_{i,j,g}(x))"] \arrow[d, Rightarrow, "p_j(F_{i,j,g}(x))"'] & h_1(\iota_i(x)) \arrow[d, Rightarrow, "p_i(x)"] \\
h_2(\iota_j(F_{i,j,g}(x))) \arrow[r, Rightarrow, "\mathsf{ap}_{h_2}(\kappa_{i,j,g}(x))"'] & h_2(\iota_i(x))
\end{tikzcd}$$

*for all $i, j : \Gamma_0$, $g : \Gamma_1(i,j)$, and $x : F_i$, then $h_1 \sim h_2$.*

*Proof.* By induction on $\mathsf{colim}_\Gamma(F)$. ☐

**Lemma 4.1.3.** *Consider a pushout square*

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{glue}"] & B \arrow[d, "\mathsf{inr}"] \\
A \arrow[r, "\mathsf{inl}"'] & A \sqcup_C B \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

*Let $Z$ be a type and $h_1, h_2 : A \sqcup_C B \to Z$. If we have terms*

$$p_1 \;:\; \prod_{a:A} h_1(\mathsf{inl}(a)) = h_2(\mathsf{inl}(a))$$
$$p_2 \;:\; \prod_{b:B} h_1(\mathsf{inr}(b)) = h_2(\mathsf{inr}(b))$$

*along with a commuting square*

$$\begin{tikzcd}
h_1(\mathsf{inl}(f(c))) \arrow[r, Rightarrow, "\mathsf{ap}_{h_1}(\mathsf{glue}(c))"] \arrow[d, Rightarrow, "p_1(f(c))"'] & h_1(\mathsf{inr}(g(c))) \arrow[d, Rightarrow, "p_2(g(c))"] \\
h_2(\mathsf{inl}(f(c))) \arrow[r, Rightarrow, "\mathsf{ap}_{h_2}(\mathsf{glue}(c))"'] & h_2(\mathsf{inr}(g(c)))
\end{tikzcd}$$

*of paths in $Z$ for every $c : C$, then $h_1 \sim h_2$.*

*Proof.* By pushout induction. ☐

Note that $\mathsf{colim}_\Gamma$ is a functor from the wild category of diagrams over $\Gamma$ to $\mathcal{U}$. In particular, for each

$(\alpha, p) : F \Rightarrow G$, the function $\mathsf{colim}(\alpha, p) : \mathsf{colim}_\Gamma(F) \to \mathsf{colim}_\Gamma(G)$ is the canonical map induced by the cocone

$$\begin{tikzcd}
F_i \arrow[rr, "F_{i,j,g}"] \arrow[d, "\alpha_i"'] & & F_j \arrow[d, "\alpha_j"] \\
G_i \arrow[rr, dotted, "G_{i,j,g}"] \arrow[dr, "\iota_i"'] & & G_j \arrow[dl, "\iota_j"] \\
& \mathsf{colim}_\Gamma(G) &
\end{tikzcd} \qquad (\lambda x.\, \mathsf{ap}_{\iota_j}(p_{i,j,g}(x))^{-1} \cdot \kappa^G_{i,j,g}(\alpha_i(x)))$$

under $F$.

Moreover, the pushout HIT is a functor on spans. For each map $(\psi, S)$

$$\begin{tikzcd}
A_1 \arrow[d, "\psi_1"'] \arrow[r, phantom, "S_1"] & C_1 \arrow[l, "f_1"'] \arrow[r, "g_1"] \arrow[d, "\psi_2"] \arrow[r, phantom, "S_2"] & B_1 \arrow[d, "\psi_3"] \\
A_2 & C_2 \arrow[l, "f_2"] \arrow[r, "g_2"'] & B_2
\end{tikzcd}$$

of spans, the function $\mathsf{po}(\psi, S) : A_1 \sqcup_{C_1} B_1 \to A_2 \sqcup_{C_2} B_2$ is the canonical map induced by the commuting square

$$\begin{tikzcd}
C_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & B_1 \arrow[d, "\mathsf{inr} \circ \psi_3"] \\
A_1 \arrow[r, "\mathsf{inl} \circ \psi_1"'] & A_2 \sqcup_{C_2} B_2
\end{tikzcd} \qquad (\lambda x.\, \mathsf{ap}_{\mathsf{inl}}(S_1(x))^{-1} \cdot \mathsf{glue}_2(\psi_2(x)) \cdot \mathsf{ap}_{\mathsf{inr}}(S_2(x)))$$

## 4.2 Colimits in coslices of $\mathcal{U}$

Let $A : \mathcal{U}$. Let $\Gamma$ be a graph and $F$ be an $A$-diagram over $\Gamma$. An $A$-cocone $(C, r, p)$ under $F$ is *colimiting* if the function

$$e_{F,T} \ : \ (C \to_A T) \to \mathsf{Cocone}_F(T)$$

$$e_{F,T}(f, f_p) \ \coloneqq \ \Big(\lambda i.\, \big(f \circ \mathsf{pr}_1(r_i), \lambda a.\, \mathsf{ap}_f(\mathsf{pr}_2(r_i)(a)) \cdot f_p(a)\big), \lambda j \lambda i \lambda g.\, \Big(\lambda x.\, \mathsf{ap}_f(\mathsf{pr}_1(p_{j,i,g})(x)), \lambda a. \Theta_{\mathsf{pr}_1(p_{j,i,g})}(f^*, a) \cdot \mathsf{ap}_{- \cdot f_p(a)}(\mathsf{ap}_{\mathsf{ap}_f}(\mathsf{pr}_2(p_{j,i,g})(a)))\Big)\Big)$$

is an equivalence for every $T : A/\mathcal{U}$, where $\Theta_{\mathsf{pr}_1(p_{i,j,g})}(f^*, a)$ has type

$$\begin{gathered}
\mathsf{ap}_f(\mathsf{pr}_1(p_{j,i,g})(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{f \circ \mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_f(\mathsf{pr}_2(r_j)(a)) \cdot f_p(a) \\
\| \\
\mathsf{ap}_f(\mathsf{pr}_1(p_{j,i,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)) \cdot f_p(a)
\end{gathered}$$

and is defined by double path induction on $\mathsf{pr}_1(p_{j,i,g})(\mathsf{pr}_2(F_i)(a))$ and $\mathsf{pr}_2(F_{i,j,g})(a)$.

**Definition 4.2.1 (Forgetful functor).** The functor $\mathsf{pr}_1 : A/\mathcal{U} \to \mathcal{U}$ induces a functor $\mathcal{F}$ :

$\mathsf{Diag}_A(\Gamma) \to \mathsf{Diag}(\Gamma)$ from the category of diagrams in $A/\mathcal{U}$ to that of diagrams in $\mathcal{U}$. It also also induces a functor $\mathcal{F} : \mathsf{Cocone}(F) \to \mathsf{Cocone}(\mathsf{pr}_1 \circ F)$ between categories of cocones for each diagram $F : \Gamma \to A/\mathcal{U}$. Specifically, $\mathcal{F}(r, K)$ maps a cocone $(C, r, K)$ under $F$ to the cocone

$$\begin{tikzcd}
\mathsf{pr}_1(F_i) \arrow[rr, "\mathsf{pr}_1(F_{i,j,g})"] \arrow[dr, "\mathsf{pr}_1(r_i)"'] & \arrow[d, phantom, "\mathsf{pr}_1(K_{i,j,g})"] & \mathsf{pr}_1(F_j) \arrow[dl, "\mathsf{pr}_1(r_j)"] \\
& \mathsf{pr}_1(C) &
\end{tikzcd}$$

under $\mathcal{F}(F)$.

Let $\mathcal{A} :\equiv (C, m, M)$ and $\mathcal{B} :\equiv (B, k, K)$ be $A$-cocones under $F$. A morphism $\mathcal{A} \to \mathcal{B}$ consists of terms

$$\begin{aligned}
f &: \mathsf{pr}_1(C) \to \mathsf{pr}_1(B) \\
p &: \prod_{a:A} f(\mathsf{pr}_2(C)(a)) = \mathsf{pr}_2(B)(a) \\
d &: \prod_{i:\Gamma_0} f \circ \mathsf{pr}_1(m_i) \sim \mathsf{pr}_1(k_i) \\
e &: \prod_{i:\Gamma_0} \prod_{a:A} d_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_f(\mathsf{pr}_2(m_i)(a)) \cdot p(a) = \mathsf{pr}_2(k_i)(a) \\
U &: \prod_{i,j,g} \prod_{x:\mathsf{pr}_1(F_i)} d_j(\mathsf{pr}_1(F_{i,j,g})(x))^{-1} \cdot \mathsf{ap}_f(\mathsf{pr}_1(M_{i,j,g})(x)) \cdot d_i(x) = \mathsf{pr}_1(K_{i,j,g})(x) \\
V &: \prod_{i,j,g} \prod_{a:A} \Lambda_{i,j,g}(a) = \mathsf{pr}_2(K_{i,j,g})(a)
\end{aligned}$$

where $\Lambda_{i,j,g}(a)$ denotes the chain of paths

$$\begin{gathered}
\mathsf{pr}_1(K_{i,j,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(k_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(k_j)(a) \\
\Big\| \scriptstyle{\textit{via } U_{i,j,g}(\mathsf{pr}_2(F_i)(a))} \\
\left(d_j(\mathsf{pr}_1(F_{i,j,g})(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_f(\mathsf{pr}_1(M_{i,j,g})(\mathsf{pr}_2(F_i)(a))) \cdot d_i(\mathsf{pr}_2(F_i)(a))\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(k_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(k_j)(a) \\
\Big\| \scriptstyle{\textit{homotopy naturality of } d_j \textit{ at } \mathsf{pr}_2(F_{i,j,g})(a)} \\
\left(\left(\mathsf{ap}_{\mathsf{pr}_1(k_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot d_j(\mathsf{pr}_2(F_j)(a))^{-1} \cdot \mathsf{ap}_{f \circ \mathsf{pr}_1(m_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\right) \cdot \mathsf{ap}_f(\mathsf{pr}_1(M_{i,j,g})(\mathsf{pr}_2(F_i)(a))) \cdot d_i(\mathsf{pr}_2(F_i)(a))\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(k_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(k_j)(a) \\
\Big\| \scriptstyle{\textit{via } e_j(a)} \\
\left(\left(\mathsf{ap}_{\mathsf{pr}_1(k_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \left(\mathsf{pr}_2(k_j)(a) \cdot \left(\mathsf{ap}_f(\mathsf{pr}_2(m_j)(a)) \cdot p(a)\right)^{-1}\right) \cdot \mathsf{ap}_{f \circ \mathsf{pr}_1(m_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\right) \cdot \mathsf{ap}_f(\mathsf{pr}_1(M_{i,j,g})(\mathsf{pr}_2(F_i)(a))) \cdot d_i(\mathsf{pr}_2(F_i)(a))\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(k_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(k_j)(a) \\
\Big\| \scriptstyle{\mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a), \mathsf{pr}_2(k_j)(a), d_i(\mathsf{pr}_2(F_i)(a)), \mathsf{pr}_1(M_{i,j,g})(\mathsf{pr}_2(F_i)(a)), \mathsf{pr}_2(m_j)(a), p(a))} \\
d_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_f(\mathsf{pr}_1(M_{i,j,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(m_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(m_j)(a)) \cdot p(a) \\
\Big\| \scriptstyle{\textit{via } \mathsf{pr}_2(M_{i,j,g})(\mathsf{pr}_2(F_i)(a))} \\
d_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_f(\mathsf{pr}_2(m_i)(a)) \cdot p(a) \\
\Big\| \scriptstyle{e_i(a)} \\
\mathsf{pr}_2(k_i)(a)
\end{gathered}$$

It's easy (albeit tedious) to equip this notion of morphism with a composition operation. Further, every $A$-cocone under $F$ has an identity morphism.

**Definition 4.2.2.** A morphism $\varphi : \mathcal{A} \to \mathcal{B}$ of $A$-cocones is an *equivalence* if $\mathsf{pr}_1(A) \xrightarrow{\mathsf{pr}_1(\varphi)} \mathsf{pr}_1(B)$ is an equivalence.

**Proposition 4.2.3.** *For each diagram $F$ in $A/\mathcal{U}$, the colimiting cocone under $F$ is unique up to unique equivalence.*

Next, we consider the interaction between colimits and reflective subuniverses of $A/\mathcal{U}$. Let $(P, \bigcirc, \eta)$ be a reflective subuniverse of $\mathcal{U}$. Then the data

$$\begin{aligned}
P_A(X) &:= P(\mathsf{pr}_1(X)) \\
\bigcirc_A(X) &:= \big(\bigcirc(\mathsf{pr}_1(X)), \eta_{\mathsf{pr}_1(X)} \circ \mathsf{pr}_2(X)\big) \\
\eta_A(X) &:= \Big(\eta_{\mathsf{pr}_1(X)}, \lambda a.\mathsf{refl}_{\eta_{\mathsf{pr}_1(X)}(\mathsf{pr}_2(X)(a))}\Big)
\end{aligned}$$

forms a reflective subuniverse of $A/\mathcal{U}$. Indeed, the two maps

$$\begin{aligned}
(\bigcirc_A X \to_A Y) &\to (X \to_A Y) \\
(f, f_p) &\mapsto \big(f \circ \eta_{\mathsf{pr}_1(X)}, f_p\big) \\
(X \to_A Y) &\to (\bigcirc_A X \to_A Y) \\
(g, g_p) &\mapsto (\mathsf{rec}_{\bigcirc_A}(g), \beta(g)(\mathsf{pr}_2(X)(a)) \cdot g_p(a))
\end{aligned}$$

are inverses of each other for all $Y$ satisfying $P(\mathsf{pr}_1(Y))$. Here, $\beta$ has type

$$\prod_{g : X \to Y} \mathsf{rec}_{\bigcirc_A}(g) \circ \eta_{\mathsf{pr}_1(X)} \sim g$$

and comes from the fact that $\eta$ is reflective.

**Lemma 4.2.4.** *Every left adjoint $A/\mathcal{U} \to A/\mathcal{U}$ preserves colimits.*

**Corollary 4.2.5.** *The functor $\bigcirc_A : A/\mathcal{U} \to A/\mathcal{U}$ creates colimits.*

## 4.3 Wedge sums in coslices

Let $A : \mathcal{U}$. Consider the data

$$\begin{aligned}
\Delta_0 &: \mathcal{U} \\
G_0 &: \Delta_0 \to A/\mathcal{U} \\
\Delta_1 &: \Delta_0 \to \Delta_0 \to \mathcal{U} \\
G &: \prod_{x, y : \Delta_0} \Delta_1(x, y) \to G_0(x) \to_A G_0(y).
\end{aligned}$$

From this data, we can construct a diagram $F$ over a graph $\Gamma$ as follows.

$$\begin{aligned}
\Gamma_0 &:= \Delta_0 + \mathbf{1} \\
F_0(\mathsf{inl}(x)) &:= \mathsf{pr}_1(G_0(x)) \\
F_0(\mathsf{inr}(*)) &:= A \\
\Gamma_1(\mathsf{inl}(x), \mathsf{inl}(y)) &:= \Delta_1(x, y) \\
\Gamma_1(\mathsf{inl}(x), \mathsf{inr}(*)) &:= \mathbf{0} \\
\Gamma_1(\mathsf{inr}(*), \mathsf{inl}(x)) &:= \mathbf{1} \\
\Gamma_1(\mathsf{inr}(*), \mathsf{inr}(*)) &:= \mathbf{0} \\
F_{\mathsf{inl}(x),\mathsf{inl}(y),\gamma} &:= \mathsf{pr}_1(G_{x,y,\gamma}) \\
F_{\mathsf{inr}(*),\mathsf{inl}(x),*} &:= \mathsf{pr}_2(G_0(x))
\end{aligned}$$

Let $\zeta(\Delta, G)$ denote this diagram.

**Lemma 4.3.1.** *Suppose that $\Delta$ is discrete. The coproduct $\bigvee_{x:\Delta} G_x$ in $A/\mathcal{U}$ fits into a commuting diagram*

$$\begin{tikzcd}
& G_i \arrow[dl, "{\left(\mathsf{inr}(i,-), \lambda a.\, \mathsf{glue}(a,i)^{-1}\right)}"'] \arrow[dr, "{\left(\iota_{\mathsf{inl}(i)}, \kappa_{\zeta(\Delta,G)}(\mathsf{inr}(*), \mathsf{inl}(i))\right)}"] & \\
\bigvee_{x:\Delta} G_x \arrow[rr, "\simeq_A"'] & & \mathsf{colim}_{x:\Gamma}\, \zeta(\Delta, G, x)
\end{tikzcd} \qquad (\text{tri-}\bigvee)$$

*under $A$.*

*Proof.* Define

$$\begin{aligned}
(\varphi, \alpha) &: \left(\bigvee_{x:\Delta} G_x\right) \to \mathsf{colim}_{x:\Gamma}\, \zeta(\Delta, G, x) \\
\varphi(\mathsf{inl}(a)) &:= \iota_{\mathsf{inr}(*)}(a) \\
\varphi(\mathsf{inr}(i, x)) &:= \iota_{\mathsf{inl}(i)}(x) \\
\mathsf{ap}_\varphi(\mathsf{glue}(a, i)) &= \kappa_{\zeta(\Delta,G)}(\mathsf{inr}(*), \mathsf{inl}(i), a)^{-1} \cdot \mathsf{ap}_{\iota_{\mathsf{inl}(i)}}(\mathsf{pr}_2(G_{\mathsf{inr}(*),\mathsf{inl}(i),*})(a)) \;:\; \iota_{\mathsf{inr}(*)}(a) = \iota_{\mathsf{inl}(i)}(\mathsf{pr}_2(G_0(i))(a)) \\
\alpha &:= \lambda a.\mathsf{refl}_{\iota_{\mathsf{inr}(*)}(a)}.
\end{aligned}$$

Conversely, define

$$\begin{aligned}
\psi &: (\mathsf{colim}_{x:\Gamma}\, \zeta(\Delta, G, x)) \to \bigvee_{x:\Delta} G_x \\
\psi(\iota_{\mathsf{inr}(*)}(a)) &:= \mathsf{inl}(a) \\
\psi(\iota_{\mathsf{inl}(i)}(x)) &:= \mathsf{inr}(i, x) \\
\mathsf{ap}_\psi(\kappa_{\zeta(\Delta,G)}(\mathsf{inr}(*), \mathsf{inl}(i), a)) &= \mathsf{ap}_{\mathsf{inr}(i,-)}(\mathsf{pr}_2(G_{\mathsf{inr}(*),\mathsf{inl}(i),*})(a)) \cdot \mathsf{glue}(a, i)^{-1} \;:\; \mathsf{inr}(i, \mathsf{pr}_1(G_{\mathsf{inr}(*),\mathsf{inl}(i),*})(a)) = \mathsf{inl}(a).
\end{aligned}$$

It is easy to prove that $\varphi$ and $\psi$ are mutual inverses as ordinary functions. By Proposition 3.3.2, it follows that $\varphi$ is an equivalence in $A/\mathcal{U}$. Moreover, it is easy to check that the triangle (tri-$\bigvee$) commutes in $A/\mathcal{U}$. $\square$

*Remark.* It is *not* the case that $\mathsf{colim}^A_\Delta G = \mathsf{colim}_{x:\Gamma}\, \zeta(\Delta, G, x)$ in general. For example, the pointed colimit of the diagram $\mathbf{1} \xrightarrow{\mathsf{id}} \mathbf{1}$ is trivial, but the colimit of the augmented diagram

$$\begin{tikzcd} & \mathbf{1} \arrow[dl, "\mathsf{id}"'] \arrow[dr, "\mathsf{id}"] & \\ \mathbf{1} \arrow[rr, "\mathsf{id}"'] & & \mathbf{1} \end{tikzcd}$$

equals $S^1$. This situtation may seem different from classical category theory, wherein colimits in coslice categories can be computed as colimits of augmented diagrams in the underlying category. Note, however, that the internal augmented diagram may add "composites" that are *not* interpreted as composites in the model of HoTT, but rather as unrelated arrows.

## 4.4 First construction of colimits in coslices

First of all, we record two variants of homotopy naturality, which will be useful for the rest of this section.

**Lemma 4.4.1.** *Let $X$ and $Y$ be types and let $f, g : X \to Y$. Suppose that $H : f \sim g$. For all $x, y : X$ and $p : x = y$,*

$$\mathsf{ap}_g(p) \;=\; H(x)^{-1} \cdot \mathsf{ap}_f(p) \cdot H(y).$$

*Proof.* By path induction on $p$. $\square$

**Lemma 4.4.2.** *Let $X$ be a type and $P : X \to \mathcal{U}$. Let $f, g : \prod_{x:X} P(x)$. For all $x, y : X$, $p : x = y$, and $H : f \sim g$, we have a commuting square*

$$\begin{tikzcd} \mathsf{transp}^P(p, f(x)) \arrow[r, Rightarrow, "\mathsf{apd}_f(p)"] \arrow[d, Rightarrow, "\mathsf{ap}_{p_*}(H(x))"'] & f(y) \arrow[d, Rightarrow, "H(y)"] \\ \mathsf{transp}^P(p, g(x)) \arrow[r, Rightarrow, "\mathsf{apd}_g(p)"'] & g(y) \end{tikzcd}$$

*Proof.* By path induction on $p$, $\square$

Let $A : \mathcal{U}$. Consider a graph $\Gamma$ and a diagram $F : \Gamma \to A/\mathcal{U}$ over $\Gamma$.

Define $\psi : \mathsf{colim}_\Gamma A \to \mathsf{colim}_\Gamma(\mathcal{F}(F))$ as the function induced by the cocone

$$\begin{tikzcd}
A \arrow[rr, "\mathsf{id}_A"] \arrow[dr, "\iota_i \circ \mathsf{pr}_2(F_i)"'] & \arrow[d, phantom, "\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))"] & A \arrow[dl, "\iota_j \circ \mathsf{pr}_2(F_j)"] \\
& \mathsf{colim}_\Gamma(\mathcal{F}(F)) &
\end{tikzcd}$$

under the constant diagram at $A$. Then form the pushout square

$$\begin{tikzcd}
\mathsf{colim}_\Gamma A \arrow[r, "\psi"] \arrow[d, "{[\mathsf{id}_A]_{i:\Gamma_0}}"'] & \mathsf{colim}_\Gamma(\mathcal{F}(F)) \arrow[d, "\mathsf{inr}"] \\
A \arrow[r, "\mathsf{inl}"'] & \mathcal{P}_A(F) \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

We can form a $A$-cocone on $(\mathcal{P}_A(F), \mathsf{inl})$ under $F$

$$\begin{tikzcd}
\mathsf{pr}_1(F_i) \arrow[rr, "F^*_{i,j,g}"] \arrow[dr, "{(\mathsf{inr} \circ \iota_i, \tau_i)}"'] & \arrow[d, phantom, "{\langle \delta_{i,j,g}, \epsilon_{i,j,g} \rangle}"] & \mathsf{pr}_1(F_j) \arrow[dl, "{(\mathsf{inr} \circ \iota_j, \tau_j)}"] \\
& \mathcal{P}_A(F) &
\end{tikzcd} \qquad (\tau_i(a) :\equiv \mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a))^{-1})$$

as follows. We have a term

$$\delta_{i,j,g} :\equiv \lambda x.\, \mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x)) \;:\; \mathsf{inr} \circ \iota_j \circ F_{i,j,g} \sim \mathsf{inr} \circ \iota_i$$

Further, for each $a : A$, we have a chain $\epsilon_{i,j,g}(a)$ of equalities

$$\begin{array}{c}
\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{inr} \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \tau_j(a) \\
\Vert \\
{\scriptstyle \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a), \tau_j(a))} \\
\Vert \\
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{refl}_{\mathsf{inl}(a)} \\
\Vert \\
{\scriptstyle \mathsf{ap}_{\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \tau_j(a) \cdot -}(\mathsf{ap}_{\mathsf{ap}_{\mathsf{inl}}}(\rho_{[\mathsf{id}_A]}(i,j,g,a)))^{-1}} \\
\Vert \\
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a))) \\
\Vert \\
{\scriptstyle \mathsf{ap}_{- \cdot \tau_j(a) \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))}(\mathsf{ap}_{\mathsf{ap}_{\mathsf{inr}}(-)^{-1}}(\rho_\psi(i,j,g,a)))^{-1}} \\
\Vert \\
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a))) \\
\Vert \\
{\scriptstyle \mathsf{PI}(\kappa_{i,j,g}(a), \tau_j(a))} \\
\Vert \\
(\kappa_{i,j,g}(a))_*(\tau_j(a)) \\
\Vert \\
{\scriptstyle \mathsf{apd}_{\mathsf{glue}(-)^{-1}}(\kappa_{i,j,g}(a))} \\
\Vert \\
\tau_i(a)
\end{array}$$

It will be convenient to decompose $\epsilon_{i,j,g}(a)$ into the following chains of paths:

1. $E_1(i,j,g,a)$, the first path of $\epsilon_{i,j,g}(a)$
2. $E_2(i,j,g,a)$, the second path of $\epsilon_{i,j,g}(a)$
3. $E_3(i,j,g,a)$, the final three paths of $\epsilon_{i,j,g}(a)$.

**Theorem 4.4.3.** *Let $(T, f_T) : A/\mathcal{U}$. The function*

$$\begin{aligned}
e_{F,T} \ &: \ (\mathcal{P}_A(F) \to_A T) \to \mathsf{lim}_{i:\Gamma^{\mathrm{op}}} (F_i \to_A T) \\
e_{F,T}(f, f_p) \ &:= \ \Big(\lambda i. \left(f \circ \mathsf{inr} \circ \iota_i, \lambda a. \, \mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\right), \lambda j \lambda i \lambda g. \left(\lambda x. \, \mathsf{ap}_f(\delta_{i,j,g}(x)), \lambda a. \Theta_{\delta_{i,j,g}}(f^*, a) \cdot \mathsf{ap}_{- \cdot f_p(a)}(\mathsf{ap}_{\mathsf{ap}_f}(\epsilon_{i,j,g}(a)))\right)\Big)
\end{aligned}$$

*is an equivalence.*

*Proof.* We define a quasi-inverse of $e_{F,T}$ as follows. Consider a cocone $(r, K) : \mathsf{lim}_{i:\Gamma^{\mathrm{op}}} (F_i \to_A T)$

$$\begin{tikzcd}
F_i \arrow[rr, "F_{i,j,g}"] \arrow[dr, "r_i"'] & \arrow[d, phantom, "K_{j,i,g}"] & F_j \arrow[dl, "r_j"] \\
& T &
\end{tikzcd}$$

under $F$. For all $i : \Gamma_0$ and $a : A$, we have that

$$\begin{aligned} & f_T(a) \\ = \; & \mathsf{pr}_1(r_i)(\mathsf{pr}_2(F_i(a))) && (\mathsf{pr}_2(r_i)(a)^{-1}) \\ \equiv \; & \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\mathsf{pr}_2(F_i(a))), \end{aligned}$$

and for all $i, j : \Gamma_0$, $g : \Gamma_1(i,j)$, and $a : A$, we have a chain $\eta_{i,j,g}(a)$ of equalities

$$\begin{gathered}
\mathsf{transp}^{x \mapsto f_T([\mathsf{id}_A](x)) = \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x))}(\kappa_{i,j,g}(a), \mathsf{pr}_2(r_j)(a)^{-1}) \\
\Vert \\
\scriptstyle \mathsf{PI}(\kappa_{i,j,g}(a), \mathsf{pr}_2(r_j)(a)) \\
\Vert \\
\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))) \\
\Vert \\
\scriptstyle \mathsf{ap}_{\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(-)}(\rho_\psi(i,j,g,a)) \\
\Vert \\
\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
\Vert \\
\scriptstyle \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a)) \\
\Vert \\
\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
\Vert \\
\scriptstyle \mathsf{ap}_{\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot _}(\rho_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(i,j,g,\mathsf{pr}_2(F_i)(a))) \\
\Vert \\
\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{pr}_1(K_{j,i,g})(\mathsf{pr}_2(F_i)(a)) \\
\Vert \\
\scriptstyle \mathsf{PI}(\mathsf{pr}_2(r_j)(a), \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)), \mathsf{pr}_1(K_{j,i,g})(\mathsf{pr}_2(F_i)(a))) \\
\Vert \\
\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \Big(\mathsf{pr}_1(K_{j,i,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)\Big)^{-1} \\
\Vert \\
\scriptstyle \mathsf{ap}_{- \cdot \left(\mathsf{pr}_1(K_{j,i,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)\right)^{-1}}(\mathsf{ap}_{\mathsf{ap}_{f_T}(-)^{-1}}(\rho_{[\mathsf{id}_A]}(i,j,g,a))) \\
\Vert \\
\Big(\mathsf{pr}_1(K_{j,i,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)\Big)^{-1} \\
\Vert \\
\scriptstyle \mathsf{ap}_{_^{-1}}(\mathsf{pr}_2(K_{j,i,g})(a)) \\
\Vert \\
\mathsf{pr}_2(r_i)(a)^{-1}
\end{gathered}$$

This gives us a function

$$\sigma \;:\; \prod_{x : \mathsf{colim}_\Gamma A} f_T([\mathsf{id}_A](x)) = \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x)) \tag{†}$$

and thus a function $h_{r,K} : \mathcal{P}_A(F) \to T$

$$
\begin{tikzcd}
\mathsf{colim}_\Gamma A \arrow[r] \arrow[d] & \mathsf{colim}_\Gamma(\mathcal{F}(F)) \arrow[d] \arrow[ddr, bend left, "\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))"] & \\
A \arrow[r] \arrow[drr, bend right, "f_T"'] & \mathcal{P}_A(F) \arrow[dr, dashed, "h_{r,K}"] & \\
 & & T
\end{tikzcd}
$$

defined by induction on $\mathcal{P}_A(F)$. Since $h(\mathsf{inl}(a)) \equiv f_T(a)$, we have a term

$$\left(h_{r,K}, \lambda a.\mathsf{refl}_{f_T(a)}\right) \ : \ \mathcal{P}_A(F) \to_A T$$

Observe that

$$
\begin{aligned}
& e_{F,T}(h_{r,K}, \lambda a.\mathsf{refl}_{f_T(a)}) \\
\equiv & \left(\lambda i.\left(\underbrace{h_{r,K} \circ \mathsf{inr} \circ \iota_i}_{\mathsf{pr}_1(r_i)}, \mathsf{ap}_{h_{r,K}}(\tau_i(a)) \cdot \mathsf{refl}_{f_T(a)}\right), \lambda j \lambda i \lambda g.\left(\lambda x.\, \mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(x)), \lambda a.\Theta_{\delta_{i,j,g}}(h^*_{r,K}, a) \cdot \mathsf{ap}_{\mathsf{ap}_{h_{r,K}}(-)\cdot \mathsf{refl}_{f_T(a)}}(\epsilon_{i,j,g}(a))\right)\right).
\end{aligned}
$$

For each $i : \Gamma_0$ and $a : A$, we have a chain $P_i(a)$ of equalities

$$
\begin{aligned}
& \mathsf{ap}_{h_{r,K}}(\tau_i(a)) \cdot \mathsf{refl}_{f_T(a)} & \\
= \ & \mathsf{ap}_{h_{r,K}}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} & (\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))) \\
= \ & \left(\mathsf{pr}_2(r_i)(a)^{-1}\right)^{-1} & (\mathsf{ap}_{-^{-1}}(\rho_{h_{r,K}}(\iota_i(a)))) \\
= \ & \mathsf{pr}_2(r_i)(a). & (\mathsf{PI}(\mathsf{pr}_2(r_i)(a)))
\end{aligned}
$$

*Notation.* We denote the path $\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))$ by $\Delta_i(a)$.

Moreover, for all $j, i : \Gamma_0$, $g : \Gamma_1(i,j)$, and $x : \mathsf{pr}_1(F_i)$, we have a chain $Q_{i,j,g}(x)$

$$
\begin{aligned}
& \mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(x)) \cdot \mathsf{refl}_{h_{r,K}(\mathsf{inr}(\iota_i(x)))} & \\
\equiv \ & \mathsf{ap}_{h_{r,K}}(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x))) \cdot \mathsf{refl}_{h_{r,K}(\mathsf{inr}(\iota_i(x)))} & \\
= \ & \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(\kappa_{i,j,g}(x)) & (\mathsf{PI}(\kappa_{i,j,g}(x))) \\
= \ & \mathsf{pr}_1(K_{j,i,g})(x). & (\rho_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(i,j,g,x))
\end{aligned}
$$

By Lemma 3.4.1, we must prove that for all $j, i : \Gamma_0$, $g : \Gamma_1(i,j)$, and $a : A$,

$$\begin{array}{c}
\mathsf{ap}_{-^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)}(Q_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \Xi\left(P, \left(\mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(x)), \Theta_{\delta_{i,j,g}}(h^*_{r,K}, a) \cdot \mathsf{ap}_{\mathsf{ap}_{h_{r,K}}(-) \cdot \mathsf{refl}_{f_T(a)}}(\epsilon_{i,j,g}(a))\right), a\right) \\
\| \\
\mathsf{pr}_2(K_{j,i,g})
\end{array}$$

To this end, note that

$$\begin{array}{c}
\Delta_i(a) \cdot \mathsf{ap}_{-^{-1}}(\rho_{h_{r,K}}(\iota_i(a))) \\
\| \\
\mathsf{transp}^{x \mapsto \mathsf{ap}_{h_{r,K}}(\mathsf{glue}_{\mathcal{P}_A(F)}(x)^{-1}) \cdot \mathsf{refl}_{f_T([\mathsf{id}_A](x))} = \sigma(x)^{-1}}(\kappa_{i,j,g}(a), \Delta_j(a) \cdot \mathsf{ap}_{-^{-1}}(\rho_{h_{r,K}}(\iota_j(a)))) \\
\| \\
\mathsf{apd}_{\mathsf{ap}_{h_{r,K}}(\mathsf{glue}_{\mathcal{P}_A(F)}(-)^{-1}) \cdot \mathsf{refl}_{f_T([\mathsf{id}_A](-))}}(\kappa_{i,j,g}(a))^{-1} \cdot \mathsf{ap}_{\mathsf{transp}^{x \mapsto \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x)) = f_T([\mathsf{id}_A](x))}(\kappa_{i,j,g}(a), -)}(\Delta_j(a) \cdot \mathsf{ap}_{-^{-1}}(\rho_{h_{r,K}}(\iota_j(a)))) \cdot \mathsf{apd}_{\sigma(-)^{-1}}(\kappa_{i,j,g}(a))
\end{array}$$

and that the triangle

$$\begin{tikzcd}
\mathsf{transp}^{x \mapsto \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x)) = f_T([\mathsf{id}_A](x))}\left(\kappa_{i,j,g}(a), \left(\mathsf{pr}_2(r_j)(a)^{-1}\right)^{-1}\right) \arrow[r, Rightarrow, "\mathsf{apd}_{\sigma(-)^{-1}}(\kappa_{i,j,g}(a))"] \arrow[d, Rightarrow, "\mathsf{Pl}(\kappa_{i,j,g}(a))"'] & \left(\mathsf{pr}_2(r_i)(a)^{-1}\right)^{-1} \\
\mathsf{transp}^{x \mapsto f_T([\mathsf{id}_A](x)) = \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x))}(\kappa_{i,j,g}(a), \mathsf{pr}_2(r_j)(a)^{-1})^{-1} \arrow[ur, Rightarrow, "\mathsf{ap}_{-^{-1}}(\mathsf{apd}_\sigma(\kappa_{i,j,g}(a)))"'] &
\end{tikzcd}$$

commutes, where $\mathsf{apd}_\sigma(\kappa_{i,j,g}(a)) = \eta_{i,j,g}(a)$ by the induction principle for $\sigma$. Therefore, after unfolding $\Xi$, we want to show that for each $a : A$, $\mathsf{pr}_2(K_{j,i,g})(a)$ equals the chain $C_\Xi(a)$, shown on the next page.

$$\mathsf{pr}_1(K_{j,i,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)$$

$$\Big\Vert \; \mathsf{ap}_{-^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)}(Q_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1}$$

$$\left(\mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \cdot \mathsf{refl}_{h_{r,K}(\mathsf{inr}(\iota_i(\mathsf{pr}_2(F_i)(a))))}\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)$$

$$\Big\Vert \; \mathsf{Pl}(\mathsf{pr}_2(F_{i,j,g})(a))$$

$$\left(\left(\mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\right) \cdot \mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \cdot \mathsf{refl}_{h_{r,K}(\mathsf{inr}(\iota_i(\mathsf{pr}_2(F_i)(a))))}\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)$$

$$\Big\Vert \; \mathsf{ap}_{\left(\left(\mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot - \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\right) \cdot \mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \cdot \mathsf{refl}_{h_{r,K}(\mathsf{inr}(\iota_i(\mathsf{pr}_2(F_i)(a))))}\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)}(\widetilde{P}_j(a))$$

$$\left(\left(\mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \left(\mathsf{pr}_2(r_j)(a) \cdot \left(\mathsf{ap}_{h_{r,K}}(\tau_j(a)) \cdot \mathsf{refl}_{f_T(a)}\right)^{-1}\right) \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\right) \cdot \mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \cdot \mathsf{refl}_{h_{r,K}(\mathsf{inr}(\iota_i(\mathsf{pr}_2(F_i)(a))))}\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)$$

$$\Big\Vert \; \mathsf{Pl}(\mathsf{pr}_2(F_{i,j,g})(a), \mathsf{pr}_2(r_j)(a), \mathsf{ap}_{h_{r,K}}(\tau_j(a)) \cdot \mathsf{refl}_{f_T(a)}, \mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(\mathsf{pr}_2(F_i)(a))))$$

$$\mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{h_{r,K}}(\tau_j(a)) \cdot \mathsf{refl}_{f_T(a)}$$

$$\Big\Vert \; \Theta_{\delta_{i,j,g}}(h^*_{r,K}, a) \cdot \mathsf{ap}_{\mathsf{ap}_{h_{r,K}}(-) \cdot \mathsf{refl}_{f_T(a)}}(\epsilon_{i,j,g}(a))$$

$$\mathsf{ap}_{h_{r,K}}(\tau_i(a)) \cdot \mathsf{refl}_{f_T(a)}$$

$$\Big\Vert \; \mathsf{apd}_{\mathsf{ap}_{h_{r,K}}(\mathsf{glue}_{\mathcal{P}_A(F)}(-)^{-1}) \cdot \mathsf{refl}_{f_T([\mathsf{id}_A](-))}}(\kappa_{i,j,g}(a))^{-1}$$

$$\mathsf{transp}^{x \mapsto \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x)) = f_T([\mathsf{id}_A](x))}(\kappa_{i,j,g}(a), \mathsf{ap}_{h_{r,K}}(\tau_j(a)) \cdot \mathsf{refl}_{f_T(a)})$$

$$\Big\Vert \; \mathsf{ap}_{\mathsf{transp}^{x \mapsto \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x)) = f_T([\mathsf{id}_A](x))}(\kappa_{i,j,g}(a), -)}(\Delta_j(a) \cdot \mathsf{ap}_{-^{-1}}(\rho_{h_{r,K}}(\iota_j(a))))$$

$$(\kappa_{i,j,g}(a))_* \left(\left(\mathsf{pr}_2(r_j)(a)^{-1}\right)^{-1}\right)$$

$$\Big\Vert \; \mathsf{Pl}(\kappa_{i,j,g}(a))$$

$$(\kappa_{i,j,g}(a))_* (\mathsf{pr}_2(r_j)(a)^{-1})^{-1}$$

$$\Big\Vert \; \mathsf{ap}_{-^{-1}}(\eta_{i,j,g}(a))$$

$$\left(\mathsf{pr}_2(r_i)(a)^{-1}\right)^{-1}$$

$$\Big\Vert \; \mathsf{Pl}(\mathsf{pr}_2(r_i)(a))$$

$$\mathsf{pr}_2(r_i)(a)$$

We can reduce $C_\Xi(a)$ to $\mathsf{pr}_2(K_{j,i,g})(a)$, which appears in $\eta_{i,j,g}(a)$, in a bottom-up fashion. This process iteratively removes the $\rho$ terms appearing in $C_\Xi(a)$. We refer the reader to the Agda formalization for the full reduction.

So far, we've defined a right inverse of $e_{F,T}$. We next want to prove that this is also a left inverse. To this end, suppose that $(f, f_p) : (\mathcal{P}_A(F) \to_A T)$ and let $E_1 :\equiv \mathsf{pr}_1(e_{F,T}(f, f_p))$ and $E_2 :\equiv \mathsf{pr}_2(e_{F,T}(f, f_p))$. We want to find terms

$$\begin{aligned}
\alpha &: \prod_{x:\mathcal{P}_A(F)} f(x) = \overbrace{h_{E_1,E_2}}^{\tilde{h}}(x) \\
\widehat{\alpha} &: \prod_{a:A} \alpha(\mathsf{inl}(a))^{-1} \cdot f_p(a) = \mathsf{refl}_{f_T(a)}
\end{aligned}$$

To construct $\alpha$, we use Lemma 4.1.3. For each $a : A$, we have that

$$\begin{aligned}
& f(\mathsf{inl}(a)) \\
&= f_T(a) && (f_p(a)) \\
&\equiv \tilde{h}(\mathsf{inl}(a)).
\end{aligned}$$

Already, we see that once $\alpha$ is constructed, it'll be easy to derive $\widehat{\alpha}$ from it. Moreover,

$$f(\mathsf{inr}(\iota_i(x))) \equiv \tilde{h}(\mathsf{inr}(\iota_i(x))),$$

We also have a chain $V_{i,j,g}(x)$ of equalities

$$\begin{array}{c}
\mathsf{transp}^{y \mapsto f(\mathsf{inr}(y)) = \tilde{h}(\mathsf{inr}(y))}(\kappa_{i,j,g}(x), \mathsf{refl}_{f(\mathsf{inr}(\iota_j(F_{i,j,g}(x))))}) \\
\Big\| \,\mathsf{PI}(\kappa_{i,j,g}(x)) \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x)))^{-1} \cdot \mathsf{ap}_{\underbrace{\tilde{h} \circ \mathsf{inr}}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}}(\kappa_{i,j,g}(x)) \\
\Big\| \,\mathsf{ap}_{\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x)))^{-1} \cdot _}(\rho_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(i,j,g,x)) \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x)))^{-1} \cdot \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x))) \\
\Big\| \,\mathsf{PI}(\kappa_{i,j,g}(x)) \\
\mathsf{refl}_{f(\mathsf{inr}(\iota_i(x)))}
\end{array}$$

By induction on $\mathsf{colim}_\Gamma(\mathcal{F}(F))$, this gives us a term

$$\gamma : \prod_{x:\mathsf{colim}_\Gamma(\mathcal{F}(F))} f(\mathsf{inr}(x)) = \tilde{h}(\mathsf{inr}(x)).$$

For all $i : \Gamma_0$ and $a : A$, we have the chain $R_i(a)$ of equalities

$$\begin{aligned}
& \Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\Big) \cdot \mathsf{ap}_{\tilde{h}}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a))) \\
= \;& \Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\Big) \cdot \big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\big)^{-1} \\
& \hfill \big(\mathsf{ap}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\right) \cdot -}(\rho_{\tilde{h}}(\iota_i(a)))\big) \\
= \;& \mathsf{refl}_{f(\mathsf{inr}(\iota_i(\mathsf{pr}_2(F_i)(a))))} \qquad \big(\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)), f_p(a))\big) \\
\equiv \;& \gamma(\psi(\iota_i(a))).
\end{aligned}$$

Further, for all $i, j : \Gamma_0$, $g : \Gamma_1(i,j)$, and $a : A$,

$$\begin{gathered}
\mathsf{transp}^{x \mapsto \left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(x))^{-1} \cdot f_p([\mathsf{id}_A](x))\right) \cdot \mathsf{ap}_{\tilde{h}}(\mathsf{glue}_{\mathcal{P}_A(F)}(x)) =_{f(\mathsf{inr}(\psi(x))) = \tilde{h}(\mathsf{inr}(\psi(x)))} \gamma(\psi(x))}(\kappa_{i,j,g}(a), R_j(a)) \\
\| \\
\mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1} \cdot f_p([\mathsf{id}_A](-))\right) \cdot \mathsf{ap}_{\tilde{h}}(\mathsf{glue}_{\mathcal{P}_A(F)}(-))}(\kappa_{i,j,g}(a))^{-1} \cdot \mathsf{ap}_{\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = \tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), -)}(R_j(a)) \cdot \mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a))
\end{gathered}$$

We must prove that this chain of equalities equals $R_i(a)$. By Lemma 4.4.2, we have a commuting square

$$\begin{tikzcd}
(\kappa_{i,j,g}(a))_* \left(\Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\Big) \cdot \mathsf{ap}_{\tilde{h}}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))\right) \arrow[r, Rightarrow, "\mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1} \cdot f_p([\mathsf{id}_A](-))\right) \cdot \mathsf{ap}_{\tilde{h}}(\mathsf{glue}_{\mathcal{P}_A(F)}(-))}(\kappa_{i,j,g}(a))"] \arrow[d, Rightarrow, "\mathsf{ap}_{\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = \tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), -)}(\mathsf{ap}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\right) \cdot -}(\rho_{\tilde{h}}(\iota_j(a))))"'] & \Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\Big) \cdot \mathsf{ap}_{\tilde{h}}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a))) \\
(\kappa_{i,j,g}(a))_* \left(\Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\Big) \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big)^{-1}\right) \arrow[r, Rightarrow, "\mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1} \cdot f_p([\mathsf{id}_A](-))\right) \cdot \sigma(-)}(\kappa_{i,j,g}(a))"'] & \Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\Big) \cdot \big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\big)^{-1} \arrow[u, Rightarrow, "\mathsf{ap}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\right) \cdot -}(\rho_{\tilde{h}}(\iota_i(a)))"']
\end{tikzcd}$$

Therefore, it suffice to show that

$$\begin{gathered}
\mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1} \cdot f_p([\mathsf{id}_A](-))\right) \cdot \sigma(-)}(\kappa_{i,j,g}(a))^{-1} \cdot \mathsf{ap}_{\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = \tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), -)}(\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)), f_p(a))) \cdot \mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a)) \\
\| \\
\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)), f_p(a))
\end{gathered}$$

where the two $\mathsf{PI}$ terms refer to those in $R_j(a)$ and $R_i(a)$, respectively.

We have commuting diagrams of paths

$$\begin{tikzcd}
 & (\kappa_{i,j,g}(a))_* \left(\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\right) \cdot \sigma(\iota_j(a))\right) \arrow[dl, Rightarrow, "\mathsf{Pl}(\kappa_{i,j,g}(a))"'] \arrow[dr, Rightarrow, "\mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1} \cdot f_p([\mathsf{id}_A](-))\right) \cdot \sigma(-)}(\kappa_{i,j,g}(a))"] & \\
\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\right) \cdot (\kappa_{i,j,g}(a))_* (\sigma(\iota_j(a))) \arrow[rr, Rightarrow, "\mathsf{ap}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\right) \cdot -}(\mathsf{apd}_\sigma(\kappa_{i,j,g}(a)))"'] & & \left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\right) \cdot \sigma(\iota_i(a))
\end{tikzcd}$$

$$\begin{tikzcd}
 & (\kappa_{i,j,g}(a))_* (\gamma(\psi(\iota_j(a)))) \arrow[dl, Rightarrow, "\mathsf{Pl}(\kappa_{i,j,g}(a))"'] \arrow[dr, Rightarrow, "\mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a))"] & \\
\mathsf{ap}_\psi(\kappa_{i,j,g}(a))_*(\gamma(\psi(\iota_j(a)))) \arrow[rr, Rightarrow, "\mathsf{apd}_\gamma(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))"'] & & \gamma(\psi(\iota_i(a)))
\end{tikzcd}$$

Note that $\mathsf{apd}_\sigma(\kappa_{i,j,g}(a)) = \eta_{i,j,g}(a)$ by the induction principle for $\mathsf{colim}_\Gamma A$. Further,

$$\begin{gathered}
\mathsf{apd}_\gamma(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))) \\
\Big\Vert \\
\mathsf{ap}_{-_*(\underbrace{\gamma(\psi(\iota_j(a)))}_{\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))}})}(\rho_\psi(i,j,g,a)) \cdot \mathsf{apd}_\gamma(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
\Big\Vert \\
\mathsf{ap}_{-_*(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})}(\rho_\psi(i,j,g,a)) \cdot \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{apd}_\gamma(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))
\end{gathered}$$

where $\mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a))$ has type

$$\Big(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))\Big)_*(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))}) \ = \ (\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))_*(\gamma(\iota_j(F_{i,j,g}(\mathsf{pr}_2(F_i)(a))))).$$

Note that $\mathsf{apd}_\gamma(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) = V_{i,j,g}(\mathsf{pr}_2(F_i)(a))$ by the induction principle for $\mathsf{colim}_\Gamma(\mathcal{F}(F))$.

Now, let

$$Y_{i,j,g}(a) \ \coloneqq \Theta_{\lambda x.\, \mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x))}(f^*, a) \cdot \mathsf{ap}_{-\cdot f_p(a)}(\mathsf{ap}_{\mathsf{ap}_f}(\epsilon_{i,j,g}(a)))$$

For each $s : f(\mathsf{inr}(\psi(\iota_j(a)))) = \tilde{h}(\mathsf{inr}(\psi(\iota_j(a))))$, consider the chain $\chi(s)$ of equalities

$$\begin{gathered}
\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = \tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), s) \\
\Big\Vert \, {\scriptstyle \mathsf{PI}(\kappa_{i,j,g}(a), s)} \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot s \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))) \\
\Big\Vert \, {\scriptstyle \mathsf{ap}_{\ldots \cdot _}(\mathsf{ap}_{\mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}}(\rho_\psi(i,j,g,a)))} \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot s \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
\Big\Vert \, {\scriptstyle \mathsf{ap}_{\ldots \cdot _}(\mu_1(i,j,g,a))} \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot s \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big) \cdot \Big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{f \circ \mathsf{inr} \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\Big)^{-1} \\
\Big\Vert \, {\scriptstyle \mathsf{ap}_{\ldots \cdot _}(\mathsf{ap}_{(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)) \cdot _^{-1}}(Y_{i,j,g}(a)))} \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot s \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big) \cdot \big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\big)^{-1} \\
\Big\Vert \, {\scriptstyle \mathsf{ap}_{\ldots \cdot _ \cdot (\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a))^{-1}}(\mu_2(i,j,g,a))} \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot s \cdot \big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))) \cdot \big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\big) \cdot \mathsf{refl}_{f_T(a)}\big) \cdot \big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\big)^{-1} \\
\Big\Vert \, {\scriptstyle \mathsf{PI}(\tau_i(a), f_p(a), \kappa_{i,j,g}(a))} \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot s \cdot \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))) \\
\Big\Vert \, {\scriptstyle \mathsf{PI}(\kappa_{i,j,g}(a), s)} \\
\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = f(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), s)
\end{gathered}$$

where $\mu_1(i,j,g,a)$ and $\mu_2(i,j,g,a)$ denote the chains of equalities

$$
\begin{gathered}
\mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
\Vert \\
{\scriptstyle \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a))} \\
\Vert \\
\mathsf{ap}_{\underbrace{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2)) \circ \iota_j}_{f \circ \mathsf{inr} \circ \iota_j}}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
\Vert \\
{\scriptstyle \mathsf{ap}_{\mathsf{ap}_{f \circ \mathsf{inr} \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \_}(\rho_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(i,j,g,\mathsf{pr}_2(F_i)(a)))} \\
\Vert \\
\mathsf{ap}_{f \circ \mathsf{inr} \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))) \\
\Vert \\
{\scriptstyle \mathsf{PI}(\tau_j(a),f_p(a))} \\
\Vert \\
\big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big) \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big)^{-1} \cdot \mathsf{ap}_{f \circ \mathsf{inr} \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))) \\
\Vert \\
{\scriptstyle \mathsf{ap}_{\left(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\right) \cdot \_}(\mathsf{PI}(\tau_j(a),f_p(a),\mathsf{pr}_2(F_{i,j,g})(a),\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))))} \\
\Vert \\
\big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big) \cdot \Big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{f \circ \mathsf{inr} \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\Big)^{-1}
\end{gathered}
$$

$$
\begin{gathered}
\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a) \\
\Vert \\
{\scriptstyle \mathsf{apd}_{\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-)^{-1}) \cdot f_p([\mathsf{id}_A](-))}(\kappa_{i,j,g}(a)^{-1})^{-1}} \\
\Vert \\
\mathsf{transp}^{y \mapsto f(\mathsf{inr}(\psi(y))) = f_T([\mathsf{id}_A](y))}(\kappa_{i,j,g}(a)^{-1}, \mathsf{ap}_f(\tau_i) \cdot f_p(a)) \\
\Vert \\
{\scriptstyle \mathsf{PI}(\kappa_{i,j,g}(a),\tau_i(a),f_p(a))} \\
\Vert \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))) \cdot \big(\mathsf{ap}_f(\tau_i) \cdot f_p(a)\big) \cdot \mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \\
\Vert \\
{\scriptstyle \mathsf{ap}_{\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))) \cdot \left(\mathsf{ap}_f(\tau_i) \cdot f_p(a)\right) \cdot \mathsf{ap}_{f_T}(-)^{-1}}(\rho_{[\mathsf{id}_A]}(i,j,g,a))} \\
\Vert \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))) \cdot \big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\big) \cdot \mathsf{refl}_{f_T(a)}
\end{gathered}
$$

respectively. By Lemma 4.4.1, we have a path

$$
\begin{gathered}
\mathsf{ap}_{\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = \tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a),-)}(\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)), f_p(a))) \\
\Vert \\
\chi(\Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\Big) \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big)^{-1}) \cdot \mathsf{ap}_{\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = f(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a),-)}(\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)), f_p(a))) \cdot \chi(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})^{-1}
\end{gathered}
$$

where $\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)), f_p(a))$ has type

$$
\Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\Big) \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big)^{-1} \;=\; \mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))}
$$

We also have the commuting square

$$\begin{tikzcd}
\Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\Big) \cdot \big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\big)^{-1} \arrow[r, Rightarrow, "\mathsf{PI}(\kappa_{i,j,g}(a))"] \arrow[d, Rightarrow, "\mathsf{ap}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\right) \cdot -}(\eta_{i,j,g}(a))^{-1}"'] & \mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = f(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), \Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\Big) \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big)^{-1}) \\
\Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\Big) \cdot \mathsf{transp}^{x \mapsto f_T([\mathsf{id}_A](x)) = \tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big)^{-1}) \arrow[r, Rightarrow, "\mathsf{PI}(\kappa_{i,j,g}(a))"'] & \mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = \tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), \Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\Big) \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big)^{-1}) \arrow[u, Rightarrow, "\chi(\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\right) \cdot \left(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\right)^{-1})"']
\end{tikzcd}$$

We've now put

$$\mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1}\cdot f_p([\mathsf{id}_A](-))\right)\cdot\sigma(-)}(\kappa_{i,j,g}(a))^{-1}\cdot\chi\left(\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1}\cdot f_p(a)\right)\cdot\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)^{-1}\right)$$

into a useful form.

Let's put $\chi(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})^{-1}\cdot\mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a))$ into a useful form as well. Consider the three chains of equalities

$$
\begin{aligned}
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\left(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))\cdot\left(\mathsf{ap}_f(\tau_i(a))\cdot f_p(a)\right)\cdot\mathsf{refl}_{f_T(a)}\right)\cdot\left(\mathsf{ap}_f(\tau_i(a))\cdot f_p(a)\right)^{-1}\\
&\qquad\Big\|\;\mathsf{ap}_{\cdots\cdot-\cdot\left(\mathsf{ap}_f(\tau_i(a))\cdot f_p(a)\right)^{-1}}(\mu_2(i,j,g,a))^{-1}\\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot\left(\mathsf{ap}_f(\tau_i(a))\cdot f_p(a)\right)^{-1}\\
&\qquad\Big\|\;\mathsf{ap}_{\cdots\cdot-}(\mathsf{ap}_{\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot-^{-1}}(\mathsf{ap}_{-\cdot f_p(a)}(\mathsf{ap}_{\mathsf{ap}_f}(E_3(i,j,g,a)))))^{-1}\\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot\left(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\tau_j(a)\cdot\mathsf{refl}_{\mathsf{inl}(a)})\cdot f_p(a)\right)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot\left(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\tau_j(a)\cdot\mathsf{refl}_{\mathsf{inl}(a)})\cdot f_p(a)\right)^{-1}\\
&\qquad\Big\|\;\mathsf{ap}_{\cdots\cdot-}(\mathsf{ap}_{\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot-^{-1}}(\mathsf{ap}_{-\cdot f_p(a)}(\mathsf{ap}_{\mathsf{ap}_f}(E_2(i,j,g,a)))))^{-1}\\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot\left(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\cdot\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1}\cdot\tau_j(a))\cdot f_p(a)\right)^{-1}\\
&\qquad\Big\|\;\mathsf{ap}_{\cdots\cdot-}(\mathsf{ap}_{\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot-^{-1}}(\mathsf{ap}_{-\cdot f_p(a)}(\mathsf{ap}_{\mathsf{ap}_f}(E_1(i,j,g,a)))))^{-1}\\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot\left(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1}\cdot\mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))\cdot\tau_j(a))\cdot f_p(a)\right)^{-1}\\
&\qquad\Big\|\;\mathsf{ap}_{\cdots\cdot-}(\mathsf{ap}_{\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot-^{-1}}(\Theta_{\lambda x.\,\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x))}(f^*,a)))^{-1}\\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot\left(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1}\cdot\mathsf{ap}_{f\circ\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))\cdot\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big) \cdot \Big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{f \circ \mathsf{inr} \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\Big)^{-1} \\
&\qquad \Big\| \; \mathsf{ap}_{\ldots \cdot \_}(\mu_1(i,j,g,a))^{-1} \\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
&\qquad \Big\| \; \mathsf{ap}_{\ldots \cdot \_}(\mathsf{ap}_{\mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}}(\rho_\psi(i,j,g,a)))^{-1} \\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))) \\
&\qquad \Big\| \; \mathsf{PI}(\kappa_{i,j,g}(a)) \\
&\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = \tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), \mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))}) \\
&\qquad \Big\| \; \mathsf{PI}(\kappa_{i,j,g}(a)) \\
&\mathsf{ap}_\psi(\kappa_{i,j,g}(a))_*(\gamma(\psi(\iota_j(a)))) \\
&\qquad \Big\| \; \mathsf{ap}_{\_{*}(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})}(\rho_\psi(i,j,g,a)) \\
&\Big(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))\Big)_* (\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))}) \\
&\qquad \Big\| \; \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a)) \\
&(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))_* (\gamma(\iota_j(F_{i,j,g}(\mathsf{pr}_2(F_i)(a))))) \\
&\qquad \Big\| \; \mathsf{PI}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
&\qquad \Big\| \; \mathsf{ap}_{\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \_}(\rho_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(i,j,g,\mathsf{pr}_2(F_i)(a))) \\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))
\end{aligned}
$$

We denote these by $P_1(i,j,g,a)$, $P_2(i,j,g,a)$, and $P_3(i,j,g,a)$, respectively. We can show that

$$
\begin{aligned}
P_1(i,j,g,a) &= \mathsf{PI}(\kappa_{i,j,g}(a), \tau_j(a), f_p(a)) \\
P_3(i,j,g,a) &= \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a), \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)), \tau_j(a), \rho_\psi(i,j,g,a))
\end{aligned}
$$

and that

$$
\begin{tikzcd}
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big) \cdot \big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{refl}_{\mathsf{inl}(a)}) \cdot f_p(a)\big)^{-1} \arrow[r, equal, "\mathsf{PI}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)){,}\tau_j(a){,}f_p(a))"] \arrow[d, Rightarrow, "P_2(i,j,g,a)"'] & \mathsf{refl}_{f(\mathsf{inr}(\iota_i(\mathsf{pr}_2(F_i)(a))))} \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big) \cdot \Big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{f\circ\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\Big)^{-1} \arrow[r, Rightarrow, "\mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a){,}\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)){,}\tau_j(a){,}\rho_\psi(i,j,g,a))"'] & \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))) \arrow[u, Rightarrow, "\mathsf{PI}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))"']
\end{tikzcd}
$$

commutes. These three equalities together give us a commuting diagram

$$
\begin{tikzcd}
\mathsf{transp}^{x\mapsto f(\mathsf{inr}(\psi(x)))=f(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), \mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))}) \arrow[rr, Rightarrow, "\mathsf{PI}(\kappa_{i,j,g}(a))"] \arrow[d, Rightarrow, "\mathsf{PI}(\kappa_{i,j,g}(a))"'] & & \mathsf{refl}_{f(\mathsf{inr}(\iota_i(\mathsf{pr}_2(F_i)(a))))} \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))) \arrow[d, Rightarrow, "\mathsf{PI}(\tau_i(a){,}f_p(a){,}\kappa_{i,j,g}(a))"'] & & \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))) \arrow[u, Rightarrow, "\mathsf{PI}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))"'] \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot \big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))) \cdot \big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\big) \cdot \mathsf{refl}_{f_T(a)}\big) \cdot \big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\big)^{-1} \arrow[dr, Rightarrow, "P_1(i,j,g,a)"'] & & \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big) \cdot \Big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{f\circ\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\Big)^{-1} \arrow[u, Rightarrow, "P_3(i,j,g,a)"'] \\
& \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big) \cdot \big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{refl}_{\mathsf{inl}(a)}) \cdot f_p(a)\big)^{-1} \arrow[ur, Rightarrow, "P_2(i,j,g,a)"'] &
\end{tikzcd}
$$

of paths.

It's easy to check that the bottom string of paths equals $\chi(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})^{-1} \cdot \mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a))$, so that

$$\chi(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})^{-1} \cdot \mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a)) \;=\; \mathsf{Pl}(\kappa_{i,j,g}(a))$$

It follows that

$$\begin{array}{c}
\mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1} \cdot f_p([\mathsf{id}_A](-))\right) \cdot \sigma(-)}(\kappa_{i,j,g}(a))^{-1} \cdot \mathsf{ap}_{\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = \tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a),-)}(\mathsf{Pl}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)), f_p(a))) \cdot \mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a)) \\
\Vert \\
\mathsf{Pl}(\kappa_{i,j,g}(a)) \cdot \mathsf{ap}_{\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = f(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a),-)}(\mathsf{Pl}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)), f_p(a))) \cdot \mathsf{Pl}(\kappa_{i,j,g}(a)) \\
\Vert \, {\scriptstyle \mathsf{Pl}(\kappa_{i,j,g}(a))} \\
\mathsf{Pl}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)), f_p(a))
\end{array}$$

as desired.

□

**Corollary 4.4.4.** *Pointed acyclic types are closed under* $\mathsf{colim}^*_\Gamma$.[1]

*Proof.* Since $\Sigma$ is a left adjoint in $\mathcal{U}^*$, we have that $\Sigma(\mathsf{colim}^*_\Gamma(F)) \simeq \mathsf{colim}^*_{i:\Gamma}(\Sigma(F_i))$. If each $F_i$ is acyclic, then the second colimit is the colimit of the constant pointed diagram at $\mathbf{1}$, which is trivial as the cofiber of the identity function on $\mathsf{colim}_\Gamma \mathbf{1}$. □

**Lemma 4.4.5.** *For every map* $h^* : T \to_A U$*, the square*

$$\begin{tikzcd}
(\mathsf{colim}\, F \to_A T) \arrow[r, "h^* \circ -"] \arrow[d, "e_{F,T}"'] & (\mathsf{colim}\, F \to_A U) \arrow[d, "e_{F,U}"] \\
\lim(F \to_A T) \arrow[r, "\lim^A_{\Gamma^{\mathrm{op}}}(f^* \circ -)"'] & \lim(F \to_A U)
\end{tikzcd}$$

*commutes where* $\lim^A_{\Gamma^{\mathrm{op}}}(f^* \circ -)$ *is defined by*

$$(x, R) \mapsto \Big(\lambda i. h^* \circ x_i, \lambda j \lambda i \lambda g. \Big(\lambda x. \mathsf{ap}_h(\mathsf{pr}_1(R_{j,i,g})(x)), \lambda a. \Theta_{\mathsf{pr}_1(R_{j,i,g})}(h^*, a) \cdot \mathsf{ap}_{\mathsf{ap}_h(-) \cdot h_p(a)}(\mathsf{pr}_2(R_{j,i,g})(a))\Big)\Big).$$

We now describe the action of $\mathsf{colim}^A_\Gamma(-)$ on morphisms. Suppose that $F$ and $G$ are $A$-diagrams over $\Gamma$. Consider a morphism $\delta :\equiv \left(d, \langle \xi, \tilde{\xi} \rangle\right)$

$$\begin{tikzcd}
F_i \arrow[r, "F_{i,j,g}"] \arrow[d, "d_i"'] \arrow[dr, phantom, "\langle \xi_{i,j,g}, \tilde{\xi}_{i,j,g} \rangle"] & F_j \arrow[d, "d_j"] \\
G_i \arrow[r, "G_{i,j,g}"'] & G_j
\end{tikzcd}$$

[1] Recall that a type is *acyclic* if its suspension is contractible (see [3]).

from $F$ to $G$. We have a commuting square

$$\begin{tikzcd}
F_i \arrow[r, "d_i"] \arrow[d, "\iota_i^F"'] & G_i \arrow[d, "\iota_i^G"] \\
\mathsf{colim}_\Gamma^A(F) \arrow[r, dashed, "\mathsf{colim}_\Gamma^A(\delta)"'] & \mathsf{colim}_\Gamma^A(G)
\end{tikzcd}$$

Indeed, we have a function $\mathsf{colim}_\Gamma(\mathcal{F}(F)) \xrightarrow{\hat{\delta}} \mathsf{colim}_\Gamma(\mathcal{F}(G))$ induced by the map

$$\begin{tikzcd}
\mathsf{pr}_1(F_i) \arrow[r, "\mathsf{pr}_1(F_{i,j,g})"] \arrow[d, "\mathsf{pr}_1(d_i)"'] \arrow[dr, phantom, "\xi_{i,j,g}"] & \mathsf{pr}_1(F_j) \arrow[d, "\mathsf{pr}_1(d_j)"] \\
\mathsf{pr}_1(G_i) \arrow[r, "\mathsf{pr}_1(G_{i,j,g})"'] & \mathsf{pr}_1(G_j)
\end{tikzcd}$$

of diagrams over $\Gamma$. Note that for each $a : A$,

$$\tilde{\xi}_{i,j,g}(a) \ : \ \xi_{i,j,g}(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a)) \cdot \mathsf{pr}_2(G_{i,j,g})(a) = \mathsf{ap}_{\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(d_j)(a)$$

Without loss of generality, we may assume that $\tilde{\xi}_{i,j,g}(a)$ instead has type

$$\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a)) = \underbrace{\mathsf{ap}_{\mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a)) \cdot \mathsf{pr}_2(G_{i,j,g})(a) \cdot \mathsf{pr}_2(d_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}}_{E_{i,j,g}(a)}$$

Now, the triangle

$$\begin{tikzcd}
& \mathsf{colim}_\Gamma A \arrow[dl, "\psi_F"'] \arrow[dr, "\psi_G"] & \\
\mathsf{colim}_\Gamma(\mathcal{F}(F)) \arrow[rr, "\hat{\delta}"'] & & \mathsf{colim}_\Gamma(\mathcal{F}(G))
\end{tikzcd}$$

commutes by induction on $\mathsf{colim}_\Gamma A$. Indeed, we have a path

$$\hat{\delta}(\psi_F(\iota_i(a))) = \hat{\delta}(\iota_i(\mathsf{pr}_2(F_i)(a))) = \iota_i(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a))) \overset{C_i(a) := \mathsf{ap}_{\iota_i}(\mathsf{pr}_2(d_i)(a))}{=\!=\!=} \iota_i(\mathsf{pr}_2(G_i)(a)) = \psi_G(\iota_i(a))$$

for all $i : \Gamma_0$ and $a : A$. By Lemma 4.4.1, we have a path

$$S_{i,j,g}(a) \ : \ \kappa_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a)) \cdot \kappa_{i,j,g}(\mathsf{pr}_2(G_i)(a)) = C_i(a)$$

Hence we have a chain $\gamma_{i,j,g}(a)$ of equalities

$$
\begin{array}{c}
(\kappa_{i,j,g}(a))_*(C_j(a)) \\
\Big\Vert\, {\scriptstyle \mathsf{PI}(\kappa_{i,j,g}(a), C_j(a))} \\
\mathsf{ap}_{\hat{\delta}}(\mathsf{ap}_{\psi_F}(\kappa_{i,j,g}(a)))^{-1} \cdot C_j(a) \cdot \mathsf{ap}_{\psi_G}(\kappa_{i,j,g}(a)) \\
\Big\Vert\, {\scriptstyle \mathsf{ap}_{\mathsf{ap}_{\hat{\delta}}(-)\cdot C_j(a)\cdot \mathsf{ap}_{\psi_G}(\kappa_{i,j,g}(a))}(\rho_{\psi_F}(i,j,g,a))} \\
\mathsf{ap}_{\hat{\delta}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot C_j(a) \cdot \mathsf{ap}_{\psi_G}(\kappa_{i,j,g}(a)) \\
\Big\Vert\, {\scriptstyle \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g}(a))} \\
\mathsf{ap}_{\hat{\delta}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot C_j(a) \cdot \mathsf{ap}_{\psi_G}(\kappa_{i,j,g}(a)) \\
\Big\Vert\, {\scriptstyle \mathsf{ap}_{\_^{-1}\cdot\ldots}(\rho_{\hat{\delta}}(i,j,g,\mathsf{pr}_2(F_i)(a)))} \\
\Big(\mathsf{ap}_{\iota_j}(\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))\Big)^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot C_j(a) \cdot \mathsf{ap}_{\psi_G}(\kappa_{i,j,g}(a)) \\
\Big\Vert\, {\scriptstyle \mathsf{ap}_{\left(\mathsf{ap}_{\iota_j}(-)^{-1}\cdot \kappa_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))\right)^{-1}\cdot\ldots}(\tilde{\xi}_{i,j,g}(a))} \\
\Big(\mathsf{ap}_{\iota_j}(E_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))\Big)^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot C_j(a) \cdot \mathsf{ap}_{\psi_G}(\kappa_{i,j,g}(a)) \\
\Big\Vert\, {\scriptstyle \mathsf{ap}_{\mathsf{ap}_{\hat{\delta}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\cdot\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1}\cdot C_j(a)\cdot\_}(\rho_{\psi_G}(i,j,g,a)))} \\
\Big(\mathsf{ap}_{\iota_j}(E_{i,j,g}(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))\Big)^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot C_j(a) \cdot \mathsf{ap}_{\iota_j}(\mathsf{pr}_2(G_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(G_i)(a)) \\
\Big\Vert\, {\scriptstyle \mathsf{PI}(\mathsf{pr}_2(d_i)(a),\mathsf{pr}_2(G_{i,j,g})(a),\mathsf{pr}_2(d_j)(a),\mathsf{pr}_2(F_{i,j,g})(a),\kappa_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a))))} \\
\kappa_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a)) \cdot \kappa_{i,j,g}(\mathsf{pr}_2(G_i)(a)) \\
\Big\Vert\, {\scriptstyle S_{i,j,g}(a)} \\
C_i(a)
\end{array}
$$

for all $i, j : \Gamma_0$, $g : \Gamma_1(i,j)$, and $a : A$. This proves that the triangle commutes.

We now have a map

$$
\begin{tikzcd}
A \arrow[d, "\mathsf{id}"'] \arrow[rr, phantom, "\mathsf{refl}_{[\mathsf{id}](x)}" near start, shift right=6ex] & & \mathsf{colim}_\Gamma A \arrow[ll] \arrow[r] \arrow[d, "\mathsf{id}"] \arrow[dr, phantom, "C^{-1}"] & \mathsf{colim}_\Gamma(\mathcal{F}(F)) \arrow[d, "\bar{\delta}"] \\
A & & \mathsf{colim}_\Gamma A \arrow[ll] \arrow[r] & \mathsf{colim}_\Gamma(\mathcal{F}(G))
\end{tikzcd}
$$

of spans. This gives us the data

$$
\begin{aligned}
\mathsf{colim}^A_\Gamma(\delta) &:\equiv \left(\Psi_\delta, \lambda a.\mathsf{refl}_{\mathsf{inl}(a)}\right) : \mathcal{P}_A(F) \to_A \mathcal{P}_A(G) \\
\Psi_\delta(\mathsf{inr}(\iota_i(x))) &\equiv \mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(x))) \\
\rho_{\Psi_\delta}(x) &: \mathsf{ap}_{\Psi_\delta}(\mathsf{glue}_{\mathcal{P}_A(F)}(x)) = \mathsf{glue}_{\mathcal{P}_A(G)}(x) \cdot \mathsf{ap}_{\mathsf{inr}}(C^{-1}(x))
\end{aligned}
$$

**Corollary 4.4.6.** *The forgetful functor $A/\mathcal{U} \to \mathcal{U}$ creates colimits over trees.*

*Proof.* Suppose that $\Gamma$ is a tree and let $F$ be an $A$-diagram over $\Gamma$. Then the function

$[\mathsf{id}_A] : \mathsf{colim}_\Gamma A \to A$ is an equivalence. One can check that

$$\begin{tikzcd}
\mathsf{colim}_\Gamma A \arrow[r, "\psi"] \arrow[d, "{[\mathsf{id}_A]}"'] & \mathsf{colim}_\Gamma(\mathcal{F}(F)) \arrow[d, "\mathsf{id}"] \\
A \arrow[r, "\psi\circ[\mathsf{id}_A]^{-1}"'] & \mathsf{colim}_\Gamma(\mathcal{F}(F))
\end{tikzcd}$$

is a pushout square. This gives us an equivalence $\gamma : \mathcal{P}_A(F) \xrightarrow{\simeq} \mathsf{colim}_\Gamma(\mathcal{F}(F))$ such that

$$\gamma(\mathsf{inr}(\iota_i(x))) \;\equiv\; \iota_i(x).$$

for all $i : \Gamma_0$ and $x : \mathsf{pr}_1(F_i)$. We also see that

$$\mathsf{ap}_\gamma(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x))) \;=\; \mathsf{ap}_{\gamma\circ\mathsf{inr}}(\kappa_{i,j,g}(x)) \;\equiv\; \mathsf{ap}_{\mathsf{id}}(\kappa_{i,j,g}(x)) \;=\; \kappa_{i,j,g}(x)$$

for all $i, j : \Gamma_0$, $g : \Gamma_1(i,j)$, and $x : \mathsf{pr}_1(F_i)$. This means that $\gamma$ is a morphism of cocones under $\mathcal{F}(F)$. It follows that the forgetful functor preserves colimits over $\Gamma$.

It remains to prove that the forgetful functor reflects colimits over $\Gamma$. Consider a cocone $\mathcal{C}$

$$\begin{tikzcd}
F_i \arrow[rr, "F_{i,j,g}"] \arrow[dr, "r_i"'] & \arrow[d, phantom, "\langle H,K\rangle"] & F_j \arrow[dl, "r_j"] \\
& C &
\end{tikzcd}$$

under $F$ as well as the cocone $\mathcal{F}(\mathcal{C}) :\equiv (\mathsf{pr}_1(C), \mathsf{pr}_1 \circ r, H)$ under $\mathcal{F}(F)$ obtained by applying the forgetful functor to $\mathcal{C}$. Suppose that $\mathcal{F}(\mathcal{C})$ is colimiting in $\mathcal{U}$. By the universal property of colimits in $A/\mathcal{U}$, we have a morphism $(\mathcal{P}_A(F), \mathsf{inl}) \xrightarrow{\tau} C$ of cocones, which induces a morphism $\mathsf{pr}_1(\mathcal{P}_A(F)) \xrightarrow{\mathcal{F}(\tau)} \mathsf{pr}_1(C)$ of cocones in $\mathcal{U}$. This morphism is unique by the universal property of colimits. Moreover, by Proposition 4.2.3, there exists a cocone equivalence from $\mathsf{pr}_1(\mathcal{P}_A(F))$ to $\mathsf{pr}_1(C)$ as both of them are colimiting. It follows that $\mathcal{F}(\tau)$ must be an equivalence. Thus, $\tau$ is a cocone morphism whose underlying function $\mathcal{P}_A(F) \to \mathsf{pr}_1(C)$ of types is an equivalence. This means that $\tau$ is a cocone equivalence, so that $\mathcal{C}$ is colimiting. ☐

**Question 4.4.7.** *Let $\Delta$ be a graph and $G$ be an $A$-diagram over $\Delta$. If the canonical function $\mathsf{colim}_\Delta(\mathcal{F}(G)) \to \mathsf{pr}_1(\mathsf{colim}^A_\Delta(G))$ is an equivalence, then is $\Delta$ a tree?*

**Corollary 4.4.8.** *If $\Gamma$ is a tree, then for each $X : A/\mathcal{U}$, the colimit $\mathsf{colim}^A_\Gamma$ of the constant diagram at $X$ is $X$.*

*Proof.* This follows easily from Corollary 2.0.4. ☐

**Note 4.4.9.** Thanks to Lemma 3.2.7, we can refine Corollary 4.4.6 as follows. If $|\Gamma|$ is $n$-connected, then so is the underlying function of the cocone morphism $\mathsf{colim}_\Gamma(\mathcal{F}(F)) \xrightarrow{\mathsf{inr}} \mathcal{P}_A(F)$ in $\mathcal{U}$. Thus, the degree to which $\mathcal{F}$ approximates $\mathsf{colim}_\Gamma^A(F)$ increases linearly with how close $\Gamma$ is to a tree.

Next, we verify that our functor $\mathsf{colim}_\Gamma^A$ is correct by showing that it's left adjoint to the constant diagram functor.

**Note 4.4.10.** Consider the $A$-cocone $K(\delta)$

$$\Big(\lambda i.\,\big(\mathsf{inr}\circ\iota_i\circ\mathsf{pr}_1(d_i),\lambda a.\,\mathsf{ap}_{\mathsf{inr}\circ\iota_i}(\mathsf{pr}_2(d_i)(a))\cdot\tau_i^G(a)\big)\,,\lambda j\lambda i\lambda g.\,\Big(\lambda x.\,\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\xi_{i,j,g}(x))^{-1}\cdot\kappa^G_{i,j,g}(\mathsf{pr}_1(d_i)(x))),\lambda a.\Theta(\epsilon_{i,j,g}(a),\tilde{\xi}_{i,j,g}(a))\Big)\Big)$$

on $\mathcal{P}_A(G)$ under $F$ where $\Theta(\epsilon_{i,j,g}(a),\tilde{\xi}_{i,j,g}(a))$ denotes the chain of equalities

$$\begin{array}{c}
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1}\cdot\kappa^G_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a))))^{-1}\cdot\mathsf{ap}_{\mathsf{inr}\circ\iota_j\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a))\cdot\mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a))\cdot\tau_j^G(a)\\
\Big\Vert\ \scriptstyle \mathsf{ap}_{\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(-)^{-1}\cdot\kappa^G_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a))))^{-1}\cdot\ldots}(\tilde{\xi}_{i,j,g}(a))\\
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(E_{i,j,g}(a))^{-1}\cdot\kappa^G_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a))))^{-1}\cdot\mathsf{ap}_{\mathsf{inr}\circ\iota_j\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a))\cdot\mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a))\cdot\tau_j^G(a)\\
\Big\Vert\ \scriptstyle \textit{via Lemma 4.4.1 applied to } \kappa^G_{i,j,g} \textit{ and } \mathsf{pr}_2(d_i)(a)\\
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(E_{i,j,g}(a))^{-1}\cdot\mathsf{ap}_{\iota_j\circ\mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a))\cdot\kappa^G_{i,j,g}(\mathsf{pr}_2(G_i)(a))\cdot\mathsf{ap}_{\iota_i}(\mathsf{pr}_2(d_i)(a))^{-1})^{-1}\cdot\mathsf{ap}_{\mathsf{inr}\circ\iota_j\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a))\cdot\mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a))\cdot\tau_j^G(a)\\
\Big\Vert\ \scriptstyle \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a),\mathsf{pr}_2(d_j)(a),\mathsf{pr}_2(d_i)(a),\kappa^G_{i,j,g}(\mathsf{pr}_2(G_i)(a)),\mathsf{pr}_2(G_{i,j,g})(a))\\
\mathsf{ap}_{\mathsf{inr}\circ\iota_i}(\mathsf{pr}_2(d_i)(a))\cdot\mathsf{ap}_{\mathsf{inr}}(\kappa^G_{i,j,g}(\mathsf{pr}_2(G_i)(a)))^{-1}\cdot\mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(G_{i,j,g})(a))\cdot\tau_j^G(a)\\
\Big\Vert\ \scriptstyle \mathsf{ap}_{\mathsf{ap}_{\mathsf{inr}\circ\iota_i}(\mathsf{pr}_2(d_i)(a))\cdot -}(\epsilon_{i,j,g}(a))\\
\mathsf{ap}_{\mathsf{inr}\circ\iota_i}(\mathsf{pr}_2(d_i)(a))\cdot\tau_i^G(a)
\end{array}$$

We have a homotopy

$$\Lambda\ :\ \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(K(\delta)))\ \sim\ \mathsf{inr}\circ\hat{\delta}$$

defined by Lemma 4.1.2 as follows.

$$\begin{aligned}
&\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(K(\delta)))(\iota_i(x))\ \equiv\ \mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(x)))\ \equiv\ \mathsf{inr}(\hat{\delta}(\iota_i(x)))\\
&\mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(K(\delta)))}(\kappa^F_{i,j,g}(x))^{-1}\cdot\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\hat{\delta}}(\kappa^F_{i,j,g}(x)))\\
= \quad &\Big(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\xi_{i,j,g}(x))^{-1}\cdot\kappa^G_{i,j,g}(\mathsf{pr}_1(d_i)(x)))\Big)^{-1}\cdot\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\hat{\delta}}(\kappa_{i,j,g}(x)))\\
&\hfill (\textit{via } \rho_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(\widetilde{K}(\delta)))}(i,j,g,x))\\
= \quad &\Big(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\xi_{i,j,g}(x))^{-1}\cdot\kappa^G_{i,j,g}(\mathsf{pr}_1(d_i)(x)))\Big)^{-1}\cdot\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\xi_{i,j,g}(x))^{-1}\cdot\kappa^G_{i,j,g}(\mathsf{pr}_1(d_i)(x)))\\
&\hfill (\textit{via } \rho_{\hat{\delta}}(i,j,g,x))\\
= \quad &\mathsf{refl}_{\mathsf{inr}(\iota_j(\mathsf{pr}_1(d_j)(F_{i,j,g}(x))))} \hfill (\mathsf{PI}(\eta_{i,j,g}(x),\kappa^G_{i,j,g}(\mathsf{pr}_1(d_i)(x))))
\end{aligned}$$

Now, let $\widetilde{K}(\delta)$ denote the same cocone with $\Theta(\epsilon_{i,j,g}(a), \tilde{\xi}_{i,j,g}(a))$ replaced by the chain

$$
\begin{gathered}
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \kappa^G_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a)) \cdot \tau^G_j(a) \\
\Big\Vert\ \mathsf{ap}_{\_^{-1}\cdot\mathsf{ap}_{\mathsf{inr}\circ\iota_j\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a))\cdot\ldots}(\rho_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(K(\delta)))}(i,j,g,\mathsf{pr}_2(F_i)(a)))^{-1} \\
\mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(K(\delta)))}(\kappa^F_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a)) \cdot \tau^G_j(a) \\
\Big\Vert\ \textit{via Lemma 4.4.1 applied to } \Lambda \textit{ and } \kappa^F_{i,j,g}(\mathsf{pr}_2(F_i)(a)) \\
\left(\overbrace{\Lambda(\iota_i(\mathsf{pr}_2(F_i)(a)))}^{\mathsf{refl}} \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\hat{\delta}}(\kappa^F_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \overbrace{\Lambda(\iota_j(F_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1}}^{\mathsf{refl}}\right) \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a)) \cdot \tau^G_j(a) \\
\Big\Vert\ \mathsf{PI}(\kappa^F_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\hat{\delta}}(\kappa^F_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a)) \cdot \tau^G_j(a) \\
\Big\Vert\ \mathsf{ap}_{\mathsf{ap}_{\mathsf{inr}}(-)^{-1}\cdot\mathsf{ap}_{\mathsf{inr}\circ\iota_j\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a))\cdot\mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a))\cdot\tau^G_j(a)}(\rho_{\hat{\delta}}(i,j,g,\mathsf{pr}_2(F_i)(a))) \\
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \kappa^G_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a)) \cdot \tau^G_j(a) \\
\Big\Vert \\
\vdots \\
\Big\Vert \\
\mathsf{ap}_{\mathsf{inr}\circ\iota_i}(\mathsf{pr}_2(d_i)(a)) \cdot \tau^G_i(a)
\end{gathered}
$$

Notice that $\widetilde{K}(\delta) = K(\delta)$.

We can use Lemma 4.1.3 to find a homotopy

$$e^{-1}_{F,\mathsf{colim}^A_\Gamma(G)}(\widetilde{K}(\delta)) \sim_A \mathsf{colim}^A_\Gamma(\delta) \ : \ \mathsf{colim}^A_\Gamma(F) \to_A \mathsf{colim}^A_\Gamma(G).$$

Indeed, we have that

$$h_{\widetilde{K}(\delta)}(\mathsf{inl}(a)) \ \equiv \ \mathsf{inl}(a) \ \equiv \ \mathsf{colim}^A_\Gamma(\delta)(\mathsf{inl}(a))$$

Additionally, we may reuse $\Lambda$ as a homotopy of type

$$\underbrace{h_{\widetilde{K}(\delta)} \circ \mathsf{inr}}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(\widetilde{K}(\delta)))} \ \sim \ \underbrace{\mathsf{colim}^A_\Gamma(\delta) \circ \mathsf{inr}}_{\mathsf{inr}\circ\hat{\delta}}$$

because $\mathcal{F}(\widetilde{K}(\delta)) \equiv \mathcal{F}(K(\delta))$. For each $i : \Gamma_0$ and $a : A$, we have a chain $\Delta(i,a)$ of paths

$$
\begin{aligned}
& \mathsf{ap}_{h_{\widetilde{K}(\delta)}}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a))) \cdot \mathsf{refl}_{\mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(\psi_F(\iota_i(a)))))} \\
= \ & \left(\mathsf{ap}_{\mathsf{inr}\circ\iota_i}(\mathsf{pr}_2(d_i)(a)) \cdot \tau^G_i(a)\right)^{-1} \cdot \mathsf{refl}_{\mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(\psi_F(\iota_i(a)))))} && (\mathsf{ap}_{-\cdot\mathsf{refl}_{\mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(\psi_F(\iota_i(a)))))}}(\rho_{h_{\widetilde{K}(\delta)}}(\iota_i(a)))) \\
= \ & \mathsf{glue}_{\mathcal{P}_A(G)}(\iota_i(a)) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_i}(\mathsf{pr}_2(d_i)(a))^{-1}) && (\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(G)}(\iota_i(a)), \mathsf{pr}_2(d_i)(a))) \\
= \ & \mathsf{ap}_{\mathsf{colim}^A_\Gamma(\delta)}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a))) && (\rho_{\Psi_\delta}(\iota_i(a))^{-1})
\end{aligned}
$$

It remains to show that

$$\begin{array}{c}\mathsf{transp}^{x \mapsto \mathsf{ap}_{h_{\underset{K(\delta)}{\sim}}}(\mathsf{glue}_{\mathcal{P}_{A}(F)}(x)) \cdot \Lambda(\psi_F(x)) = \mathsf{ap}_{\Psi_\delta}(\mathsf{glue}_{\mathcal{P}_{A}(F)}(x))}(\kappa_{i,j,g}(a), \Delta(j,a)) \\ \| \\ \Delta(i,a)\end{array}$$

We have an equality

$$\begin{array}{c}\mathsf{transp}^{x \mapsto \mathsf{ap}_{h_{\underset{K(\delta)}{\sim}}}(\mathsf{glue}_{\mathcal{P}_{A}(F)}(x)) \cdot \Lambda(\psi_F(x)) = \mathsf{ap}_{\Psi_\delta}(\mathsf{glue}_{\mathcal{P}_{A}(F)}(x))}(\kappa_{i,j,g}(a), \Delta(j,a)) \\ \| \\ \mathsf{apd}_{\mathsf{ap}_{h_{\underset{K(\delta)}{\sim}}}(\mathsf{glue}_{\mathcal{P}_{A}(F)}(-)) \cdot \Lambda(\psi_F(-))}(\kappa_{i,j,g}(a))^{-1} \cdot \mathsf{ap}_{\mathsf{transp}^{x \mapsto \mathsf{inl}([\mathsf{id}_A](x)) = \mathsf{inr}(\hat{\delta}(\psi_F(x)))}(\kappa_{i,j,g}(a), -)}(\Delta(j,a)) \cdot \mathsf{apd}_{\mathsf{ap}_{\Psi_\delta}(\mathsf{glue}_{\mathcal{P}_{A}(F)}(-))}(\kappa_{i,j,g}(a))\end{array}$$

along with commuting diagrams shown on the next page, where $\sigma$ was defined at (†).

$$
\begin{tikzcd}
(\kappa_{i,j,g}(a))_*\left(\mathsf{ap}_{h_{\widetilde{K(\delta)}}}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))\cdot \mathsf{refl}_{\mathsf{inr}(\iota_j(\mathsf{pr}_1(d_j)(\psi(\iota_j(a)))))}\right) \arrow[rr, Rightarrow, "\mathsf{apd}_{\mathsf{ap}_{h_{\widetilde{K(\delta)}}}(\mathsf{glue}_{\mathcal{P}_A(F)}(-))\cdot\Lambda(\psi_F(-))}(\kappa_{i,j,g}(a))"] \arrow[d, Rightarrow, "\mathsf{ap}_{(\kappa_{i,j,g}(a))_*}(\mathsf{ap}_{-\cdot\mathsf{refl}_{\mathsf{inr}(\iota_j(\mathsf{pr}_1(d_j)(\psi(\iota_j(a)))))}}(\rho_{h_{\widetilde{K(\delta)}}}(\iota_j(a))))"] & & \mathsf{ap}_{h_{\widetilde{K(\delta)}}}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))\cdot \mathsf{refl}_{\mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(\psi(\iota_i(a)))))} \arrow[d, Rightarrow, "\mathsf{ap}_{-\cdot\mathsf{refl}_{\mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(\psi(\iota_i(a)))))}}(\rho_{h_{\widetilde{K(\delta)}}}(\iota_i(a)))"] \\
(\kappa_{i,j,g}(a))_*\left(\left(\mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a))\cdot\tau_j^G(a)\right)^{-1}\cdot \mathsf{refl}_{\mathsf{inr}(\iota_j(\mathsf{pr}_1(d_j)(\psi(\iota_j(a)))))}\right) \arrow[rr, Rightarrow, "\mathsf{apd}_{\sigma(-)\cdot\Lambda(\psi_F(-))}(\kappa_{i,j,g}(a))"'] \arrow[dr, Rightarrow, "\mathsf{PI}(\kappa_{i,j,g}(a))"'] & & \left(\mathsf{ap}_{\mathsf{inr}\circ\iota_i}(\mathsf{pr}_2(d_i)(a))\cdot\tau_i^G(a)\right)^{-1}\cdot \mathsf{refl}_{\mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(\psi(\iota_i(a)))))} \\
& (\kappa_{i,j,g}(a))_*\left(\left(\mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a))\cdot\tau_j^G(a)\right)^{-1}\right)\cdot \mathsf{refl}_{\mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(\psi(\iota_i(a)))))} \arrow[ur, Rightarrow, "\mathsf{ap}_{-\cdot\mathsf{refl}_{\mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(\psi(\iota_i(a)))))}}(\mathsf{apd}_\sigma(\kappa_{i,j,g}(a)))"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\kappa_{i,j,g}(a))_*(\mathsf{ap}_{\Psi_\delta}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))) \arrow[rr, Rightarrow, "\mathsf{apd}_{\mathsf{ap}_{\Psi_\delta}(\mathsf{glue}_{\mathcal{P}_A(F)}(-))}(\kappa_{i,j,g}(a))"] \arrow[d, Rightarrow, "\mathsf{ap}_{(\kappa_{i,j,g}(a))_*}(\rho_{\Psi_\delta}(\iota_j(a)))"] & & \mathsf{ap}_{\Psi_\delta}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a))) \arrow[d, Rightarrow, "\rho_{\Psi_\delta}(\iota_i(a))"] \\
(\kappa_{i,j,g}(a))_*(\mathsf{glue}_{\mathcal{P}_A(G)}(\iota_j(a))\cdot\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(d_j)(a))^{-1})) \arrow[rr, Rightarrow, "\mathsf{apd}_{\mathsf{glue}_{\mathcal{P}_A(G)}(-)\cdot\mathsf{ap}_{\mathsf{inr}}(C^{-1}(-))}(\kappa_{i,j,g}(a))"'] \arrow[dr, Rightarrow, "\mathsf{PI}(\kappa_{i,j,g}(a))"'] & & \mathsf{glue}_{\mathcal{P}_A(G)}(\iota_i(a))\cdot\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_i}(\mathsf{pr}_2(d_i)(a))^{-1}) \\
& \mathsf{glue}_{\mathcal{P}_A(G)}(\iota_i(a))\cdot\mathsf{ap}_{\mathsf{inr}}((\kappa_{i,j,g}(a))_*(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(d_j)(a)))^{-1}) \arrow[ur, Rightarrow, "\mathsf{ap}_{\mathsf{glue}_{\mathcal{P}_A(G)}(\iota_i(a))\cdot\mathsf{ap}_{\mathsf{inr}}(-^{-1})}(\mathsf{apd}_C(\kappa_{i,j,g}(a)))"'] &
\end{tikzcd}
$$

These two commuting diagrams are due in part to Lemma 4.4.2. We have that

$$
\begin{aligned}
\mathsf{apd}_{\sigma}(\kappa_{i,j,g}(a)) &= \eta_{i,j,g}(a) && (\rho_{\sigma}(i,j,g,a)) \\
\mathsf{apd}_{C}(\kappa_{i,j,g}(a)) &= \gamma_{i,j,g}(a) && (\rho_{C}(i,j,g,a))
\end{aligned}
$$

Thus, it suffices to prove that the diagram

$$
\begin{tikzcd}
\left(\mathsf{ap}_{\mathsf{inr}\circ\iota_i}(\mathsf{pr}_2(d_i)(a)) \cdot \tau_i^G(a)\right)^{-1} \cdot \mathsf{refl}_{\mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(\psi(\iota_i(a)))))} \arrow[r, Rightarrow, "\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(G)}(\iota_i(a)){,}\mathsf{pr}_2(d_i)(a))"] \arrow[d, Rightarrow, "\mathsf{ap}_{-\cdot\mathsf{refl}_{\mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(\psi(\iota_i(a)))))}}(\eta_{i,j,g}(a))^{-1}"'] & \mathsf{glue}_{\mathcal{P}_A(G)}(\iota_i(a)) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_i}(\mathsf{pr}_2(d_i)(a))^{-1}) \\
(\kappa_{i,j,g}(a))_*\left(\left(\mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a)) \cdot \tau_j^G(a)\right)^{-1}\right) \cdot \mathsf{refl}_{\mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(\psi(\iota_i(a)))))} \arrow[d, Rightarrow, "\mathsf{PI}(\kappa_{i,j,g}(a))"'] & \mathsf{glue}_{\mathcal{P}_A(G)}(\iota_i(a)) \cdot \mathsf{ap}_{\mathsf{inr}}((\kappa_{i,j,g}(a))_*(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(d_j)(a)))^{-1}) \arrow[u, Rightarrow, "\mathsf{ap}_{\mathsf{glue}_{\mathcal{P}_A(G)}(\iota_i(a))\cdot\mathsf{ap}_{\mathsf{inr}}(-^{-1})}(\gamma_{i,j,g}(a))"'] \\
(\kappa_{i,j,g}(a))_*\left(\left(\mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a)) \cdot \tau_j^G(a)\right)^{-1} \cdot \mathsf{refl}_{\mathsf{inr}(\iota_j(\mathsf{pr}_1(d_j)(\psi(\iota_j(a)))))}\right) \arrow[r, Rightarrow, "\mathsf{ap}_{\mathsf{transp}^{x\mapsto\mathsf{inl}([\mathsf{id}_A](x))=\mathsf{inr}(\hat{\delta}(\psi_F(x)))}(\kappa_{i,j,g}(a),-)}(\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(G)}(\iota_j(a)){,}\mathsf{pr}_2(d_j)(a)))"'] & (\kappa_{i,j,g}(a))_*(\mathsf{glue}_{\mathcal{P}_A(G)}(\iota_j(a)) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(d_j)(a))^{-1})) \arrow[u, Rightarrow, "\mathsf{PI}(\kappa_{i,j,g}(a))"']
\end{tikzcd}
$$

commutes. We leave the proof, which resembles the proof of Theorem 4.4.3, to the Agda formalization.

**Definition 4.4.11.** Define $\mathsf{lim}(-\circ\delta) : \mathsf{lim}(G \to_A T) \to \mathsf{lim}(F \to_A T)$ by

$$
(x, \langle R_1, R_2\rangle) \;\mapsto\; \Big(\lambda i.\Big(\mathsf{pr}_1(x_i)\circ\mathsf{pr}_1(d_i), \lambda a.\, \mathsf{ap}_{\mathsf{pr}_1(x_i)}(\mathsf{pr}_2(d_i)(a)) \cdot \mathsf{pr}_2(x_i)(a)\Big), \lambda j\lambda i\lambda g.\Big(\lambda x.\, \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\xi_{i,j,g}(x))^{-1} \cdot R_1(j,i,g,\mathsf{pr}_1(d_i)(x)), \lambda a. V_{x,R_1,R_2}(j,i,g,a)\Big)\Big)
$$

where $V_{x,R_1,R_2}(j,i,g,a)$ denotes the chain of equalities

$$
\begin{array}{c}
\left(\mathsf{ap}_{\mathsf{pr}_1(x_j)}(\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot R_1(j,i,g,\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{pr}_2(d_j)(a)) \cdot \mathsf{pr}_2(x_j)(a) \\
\Vert \\
\mathsf{PI}(R_1(j,i,g,\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a))),\mathsf{ap}_{\mathsf{pr}_1(x_j)}(\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a)))) \\
\Vert \\
R_1(j,i,g,\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{pr}_2(d_j)(a)) \cdot \mathsf{pr}_2(x_j)(a) \\
\Vert \\
\mathsf{ap}_{R_1(j,i,g,\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))^{-1}\cdot\mathsf{ap}_{\mathsf{pr}_1(x_j)}(-)\cdot\mathsf{ap}_{\mathsf{pr}_1(x_j)\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a))\cdot\mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{pr}_2(d_j)(a))\cdot\mathsf{pr}_2(x_j)(a)}(\tilde{\xi}_{i,j,g}(a)) \\
\Vert \\
R_1(j,i,g,\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(E_{i,j,g}(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{pr}_2(d_j)(a)) \cdot \mathsf{pr}_2(x_j)(a) \\
\Vert \\
\textit{via Lemma 4.4.1 applied to } R_1 \textit{ and } \mathsf{pr}_2(d_i)(a) \\
\Vert \\
\left(\mathsf{ap}_{\mathsf{pr}_1(x_i)}(\mathsf{pr}_2(d_i)(a)) \cdot R_1(j,i,g,\mathsf{pr}_2(G_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{ap}_{\mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a)))^{-1}\right) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(E_{i,j,g}(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{pr}_2(d_j)(a)) \cdot \mathsf{pr}_2(x_j)(a) \\
\Vert \\
\mathsf{PI}(\mathsf{ap}_{\mathsf{pr}_1(x_i)}(\mathsf{pr}_2(d_i)(a)),\mathsf{ap}_{\mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a)),\mathsf{pr}_2(G_{i,j,g})(a),\mathsf{pr}_2(d_j)(a),\mathsf{pr}_2(F_{i,j,g})(a),R_1(j,i,g,\mathsf{pr}_2(G_i)(a))^{-1}) \\
\Vert \\
\mathsf{ap}_{\mathsf{pr}_1(x_i)}(\mathsf{pr}_2(d_i)(a)) \cdot R_1(j,i,g,\mathsf{pr}_2(G_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{pr}_2(G_{i,j,g})(a)) \cdot \mathsf{pr}_2(x_j)(a) \\
\Vert \\
\mathsf{ap}_{\mathsf{ap}_{\mathsf{pr}_1(x_i)}(\mathsf{pr}_2(d_i)(a))\cdot -}(R_2(j,i,g,a)) \\
\Vert \\
\mathsf{ap}_{\mathsf{pr}_1(x_i)}(\mathsf{pr}_2(d_i)(a)) \cdot \mathsf{pr}_2(x_i)(a)
\end{array}
$$

**Lemma 4.4.12.** *The square*

$$\begin{tikzcd}
(\mathsf{colim}(G) \to_A T) \arrow[r, "- \circ \mathsf{colim}^A_\Gamma(\delta)"] \arrow[d, "e_{G,T}"'] & (\mathsf{colim}(F) \to_A T) \arrow[d, "e_{F,T}"] \\
\mathsf{lim}(G \to_A T) \arrow[r, "\mathsf{lim}^A_{\Gamma^{\mathrm{op}}}(- \circ \delta)"'] & \mathsf{lim}(F \to_A T)
\end{tikzcd}$$

*commutes.*

*Proof.* For each $f^* : \mathsf{colim}(G) \to_A T$, note that

$$\begin{aligned}
& e_{F,T}(f^* \circ e^{-1}_{F,\mathsf{colim}(G)}(K(\delta))) \\
= \;& \mathsf{lim}^A_{\Gamma^{\mathrm{op}}}(f^* \circ -)(e_{F,\mathsf{colim}(G)}(e^{-1}_{F,\mathsf{colim}(G)}(K(\delta)))) && \text{(Lemma 4.4.5)} \\
= \;& \mathsf{lim}^A_{\Gamma^{\mathrm{op}}}(f^* \circ -)(K(\delta)).
\end{aligned}$$

Thus, it suffices to prove that

$$\mathsf{lim}^A_{\Gamma^{\mathrm{op}}}(f^* \circ -)(K(\delta)) \; = \; \mathsf{lim}^A_{\Gamma^{\mathrm{op}}}(- \circ \delta)(e_{G,T}(f^*))$$

We leave such a proof, which is messy yet routine, to the Agda formalization. □

**Corollary 4.4.13.** *We have an adjunction* $\mathsf{colim}^A_\Gamma \dashv \mathsf{const}_\Gamma$*, where* $\mathsf{const}_\Gamma$ *denotes the constant diagram functor* $A/\mathcal{U} \to \mathsf{Diag}_A(\Gamma)$*.*

## 4.5 Second construction of colimits in coslices

In this section, we apply the $3 \times 3$ lemma to our first construction of $\mathsf{colim}^A_\Gamma(F)$ to obtain the familiar construction of $\mathsf{colim}^A_\Gamma(F)$ as a pushout of coproducts in $A/\mathcal{U}$.

To begin, consider the following grid of commuting squares:

$$\begin{tikzcd}
\sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)} \mathsf{pr}_1(F_i) & \left(\sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)} \mathsf{pr}_1(F_i)\right) + \left(\sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)} \mathsf{pr}_1(F_i)\right) \arrow[l, "\mathsf{id} + \mathsf{id}"'] \arrow[r, "(i,x) + (j,\mathsf{pr}_1(F_{i,j,g})(x))"] & \sum_{i:\Gamma_0} \mathsf{pr}_1(F_i) \\
\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right) \times A \arrow[u, "{(i,j,g,\mathsf{pr}_2(F_i)(a))}"] \arrow[d, "\mathsf{pr}_2"'] \arrow[ur, phantom, "\mathsf{refl}_{(i,j,g,\mathsf{pr}_2(F_i)(a))} + \mathsf{refl}_{(i,j,g,\mathsf{pr}_2(F_i)(a))}"] & \left(\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right) \times A\right) + \left(\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right) \times A\right) \arrow[l, "\mathsf{id} + \mathsf{id}"'] \arrow[r, "(i,a) + (j,a)"] \arrow[u, "{(i,j,g,\mathsf{pr}_2(F_i)(a)) + (i,j,g,\mathsf{pr}_2(F_i)(a))}"] \arrow[d, "\mathsf{pr}_2 + \mathsf{pr}_2"] \arrow[ur, phantom, "\mathsf{refl}_{(i,\mathsf{pr}_2(F_i)(a))} + \mathsf{ap}_{(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a))"] & \Gamma_0 \times A \arrow[u, "{(i,\mathsf{pr}_2(F_i)(a))}"'] \arrow[d, "\mathsf{pr}_2"'] \\
A \arrow[ur, phantom, "\mathsf{refl}_a + \mathsf{refl}_a"] & A \arrow[l, "\mathsf{id}_A"] \arrow[r, "\mathsf{id}_A"'] \arrow[ur, phantom, "\mathsf{refl}_a + \mathsf{refl}_a"] & A
\end{tikzcd}$$

Call the pushouts of the left, middle, and right vertical spans $V_1$, $V_2$, and $V_3$, respectively. Call the pushouts of the top, middle, and bottom horizontal spans $H_1$, $H_2$, and $H_3$, respectively. We can form two additional pushouts

$$
\begin{tikzcd}
V_2 \arrow[r, "\delta_2"] \arrow[d, "\delta_1"'] & V_3 \arrow[d] \\
V_1 \arrow[r] & P_V \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
H_2 \arrow[r, "\eta_1"] \arrow[d, "\eta_2"'] & H_1 \arrow[d] \\
H_3 \arrow[r] & P_H \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

where

- $\delta_1$ denotes the function induced by the middle-to-left map of spans;
- $\delta_2$ the function induced by the middle-to-right map of spans;
- $\eta_1$ the function induced by the middle-to-top map of spans; and
- $\eta_2$ the function induced by the middle-to-bottom map of spans.

Licata and Brunerie construct an equivalence $\tau_1 : P_H \xrightarrow{\simeq} P_V$ of types by double induction on pushouts [6, Section VII], which satisfies, in particular,

$$
\begin{aligned}
\tau_1(\mathsf{inl}(\mathsf{inl}(a))) \;&\equiv\; \mathsf{inl}(\mathsf{inl}(a)) \\
\tau_1(\mathsf{inr}(\mathsf{inr}(i,x))) \;&\equiv\; \mathsf{inr}(\mathsf{inr}(i,x)).
\end{aligned}
$$

**Lemma 4.5.1.** *We have an equivalence*

$$
\xi \;:\; V_2 \xrightarrow{\simeq} \left(\bigvee_{i,j,g} \mathsf{pr}_1(F_i)\right) \vee \left(\bigvee_{i,j,g} \mathsf{pr}_1(F_i)\right)
$$

*Proof.* Define $\xi$ by recursion on the commuting square

$$
\begin{tikzcd}
\left(\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right)\times A\right) + \left(\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right)\times A\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}_{\bigvee_{i,j,g}\mathsf{pr}_1(F_i)})(i,j,g,a) \;+\; \mathsf{glue}_{\bigvee\vee\bigvee}(a)\cdot\mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}_{\bigvee_{i,j,g}\mathsf{pr}_1(F_i)})(i,j,g,a)"] & \left(\sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)}\mathsf{pr}_1(F_i)\right) + \left(\sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)}\mathsf{pr}_1(F_i)\right) \arrow[d, "\mathsf{inl}\circ\mathsf{inr} \;+\; \mathsf{inr}\circ\mathsf{inr}"] \\
A \arrow[r, "a\mapsto\mathsf{inl}(\mathsf{inl}(a))"'] & \left(\bigvee_{i,j,g}\mathsf{pr}_1(F_i)\right)\vee\left(\bigvee_{i,j,g}\mathsf{pr}_1(F_i)\right)
\end{tikzcd}
$$

Define a quasi-inverse $\tilde{\xi}$ of $\xi$ by recursion on $\bigvee\vee\bigvee$ with the commuting square

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathsf{refl}_{\mathsf{inl}(a)}"] & \bigvee_{i,j,g}\mathsf{pr}_1(F_i) \arrow[d, "\epsilon_2"] \\
\bigvee_{i,j,g}\mathsf{pr}_1(F_i) \arrow[r, "\epsilon_1"'] & V_2
\end{tikzcd}
$$

Here, $\epsilon_1$ and $\epsilon_2$ are induced by the commuting squares

$$\begin{tikzcd}
\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right)\times A \arrow[r] \arrow[d] \arrow[dr, phantom, "{(i,j,g,a)\mapsto \mathsf{glue}_{V_2}(\mathsf{inl}(i,j,g,a))}"] & \sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)}\mathsf{pr}_1(F_i) \arrow[d, "t\mapsto \mathsf{inr}(\mathsf{inl}(t))"] \\
A \arrow[r, "a\mapsto \mathsf{inl}(a)"'] & V_2
\end{tikzcd}$$

$$\begin{tikzcd}
\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right)\times A \arrow[r] \arrow[d] \arrow[dr, phantom, "{(i,j,g,a)\mapsto \mathsf{glue}_{V_2}(\mathsf{inr}(i,j,g,a))}"] & \sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)}\mathsf{pr}_1(F_i) \arrow[d, "t\mapsto \mathsf{inr}(\mathsf{inr}(t))"] \\
A \arrow[r, "a\mapsto \mathsf{inl}(a)"'] & V_2
\end{tikzcd}$$

respectively. By induction on $V_2$, we prove that $\tilde{\xi}\circ\xi\sim \mathsf{id}_{V_2}$. We have that

$$\begin{aligned}
\tilde{\xi}(\xi(\mathsf{inl}(a))) &\equiv \tilde{\xi}(\mathsf{inl}(\mathsf{inl}(a))) \\
&\equiv \epsilon_1(\mathsf{inl}(a)) \\
&\equiv \mathsf{inl}(a) \\
\tilde{\xi}(\xi(\mathsf{inr}(\mathsf{inl}(i,j,g,x)))) &\equiv \tilde{\xi}(\mathsf{inl}(\mathsf{inr}(i,j,g,x))) \\
&\equiv \epsilon_1(\mathsf{inr}(i,j,g,x)) \\
&\equiv \mathsf{inr}(\mathsf{inl}(i,j,g,x)) \\
\tilde{\xi}(\xi(\mathsf{inr}(\mathsf{inr}(i,j,g,x)))) &\equiv \tilde{\xi}(\mathsf{inr}(\mathsf{inr}(i,j,g,x))) \\
&\equiv \epsilon_2(\mathsf{inr}(i,j,g,x)) \\
&\equiv \mathsf{inr}(\mathsf{inr}(i,j,g,x)) \\
\mathsf{transp}^{x\mapsto\tilde{\xi}(\xi(x))=x}(\mathsf{glue}_{V_2}(\mathsf{inl}(i,j,g,a)),\mathsf{refl}_{\mathsf{inl}(a)}) &= \mathsf{ap}_{\tilde{\xi}}(\mathsf{ap}_{\xi}(\mathsf{glue}(\mathsf{inl}(i,j,g,a))))^{-1}\cdot\mathsf{glue}(\mathsf{inl}(i,j,g,a)) \\
&= \mathsf{glue}(\mathsf{inl}(i,j,g,a))^{-1}\cdot\mathsf{glue}(\mathsf{inl}(i,j,g,a)) \\
&= \mathsf{refl}_{\mathsf{inr}(\mathsf{inl}(i,j,g,\mathsf{pr}_2(F_i)(a)))} \\
\mathsf{transp}^{x\mapsto\tilde{\xi}(\xi(x))=x}(\mathsf{glue}_{V_2}(\mathsf{inr}(i,j,g,a)),\mathsf{refl}_{\mathsf{inl}(a)}) &= \mathsf{ap}_{\tilde{\xi}}(\mathsf{ap}_{\xi}(\mathsf{glue}(\mathsf{inr}(i,j,g,a))))^{-1}\cdot\mathsf{glue}(\mathsf{inr}(i,j,g,a)) \\
&= \mathsf{glue}(\mathsf{inr}(i,j,g,a))^{-1}\cdot\mathsf{glue}(\mathsf{inr}(i,j,g,a)) \\
&= \mathsf{refl}_{\mathsf{inr}(\mathsf{inr}(i,j,g,\mathsf{pr}_2(F_i)(a)))}
\end{aligned}$$

Conversely, we get a homotopy $\xi \circ \tilde{\xi} \sim \mathsf{id}_{\bigvee \vee \bigvee}$ from induction on $\bigvee \vee \bigvee$ as follows.

$$\begin{aligned}
\xi(\tilde{\xi}(\mathsf{inl}(\mathsf{inr}(i,j,g,x)))) &\equiv \xi(\mathsf{inr}(\mathsf{inl}(i,j,g,x))) \\
&\equiv \mathsf{inl}(\mathsf{inr}(i,j,g,x)) \\
\xi(\tilde{\xi}(\mathsf{inl}(\mathsf{inl}(a)))) &\equiv \xi(\mathsf{inl}(a)) \\
&\equiv \mathsf{inl}(\mathsf{inl}(a)) \\
\mathsf{transp}^{x \mapsto \xi(\tilde{\xi}(\mathsf{inl}(x)))=\mathsf{inl}(x)}(\mathsf{glue}_{\bigvee_{i,j,g} \mathsf{pr}_1(F_i)}(i,j,g,a), \mathsf{refl}_{\mathsf{inl}(a)}) &= \mathsf{ap}_{\xi}(\mathsf{ap}_{\underbrace{\tilde{\xi} \circ \mathsf{inl}}_{\epsilon_1}}(\mathsf{glue}(i,j,g,a)))^{-1} \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}(i,j,g,a)) \\
&= \mathsf{ap}_{\xi}(\mathsf{glue}_{V_2}(\mathsf{inl}(i,j,g,a)))^{-1} \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}(i,j,g,a)) \\
&\quad (\mathsf{ap}_{-^{-1} \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}(i,j,g,a))}(\mathsf{ap}_{\mathsf{ap}_{\xi}}(\rho_{\epsilon_1}(i,j,g,a)))) \\
&= \mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}(i,j,g,a))^{-1} \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}(i,j,g,a)) \\
&\qquad (\mathsf{ap}_{-^{-1} \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}(i,j,g,a))}(\rho_{\xi}(\mathsf{inl}(i,j,g,a)))) \\
&= \mathsf{refl}_{\mathsf{inr}(\mathsf{inl}(i,j,g,\mathsf{pr}_2(F_i)(a)))} \\
\xi(\tilde{\xi}(\mathsf{inr}(\mathsf{inr}(i,j,g,x)))) &\equiv \xi(\mathsf{inr}(\mathsf{inr}(i,j,g,x))) \\
&\equiv \mathsf{inr}(\mathsf{inr}(i,j,g,x)) \\
\xi(\tilde{\xi}(\mathsf{inr}(\mathsf{inl}(a)))) &\equiv \xi(\mathsf{inl}(a)) \\
&\equiv \mathsf{inl}(\mathsf{inl}(a)) \\
&= \mathsf{inr}(\mathsf{inl}(a)) \qquad\qquad (\mathsf{glue}_{\bigvee \vee \bigvee}(a)) \\
\mathsf{transp}^{x \mapsto \xi(\tilde{\xi}(\mathsf{inr}(x)))=\mathsf{inr}(x)}(\mathsf{glue}_{\bigvee_{i,j,g} \mathsf{pr}_1(F_i)}(i,j,g,a), \mathsf{glue}(a)) &= \mathsf{ap}_{\xi}(\mathsf{ap}_{\underbrace{\tilde{\xi} \circ \mathsf{inr}}_{\epsilon_2}}(\mathsf{glue}(i,j,g,a)))^{-1} \cdot \mathsf{glue}(a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}(i,j,g,a)) \\
&= \mathsf{ap}_{\xi}(\mathsf{glue}_{V_2}(\mathsf{inr}(i,j,g,a)))^{-1} \cdot \mathsf{glue}(a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}(i,j,g,a)) \\
&\quad (\mathsf{ap}_{-^{-1} \cdot \mathsf{glue}(a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}(i,j,g,a))}(\mathsf{ap}_{\mathsf{ap}_{\xi}}(\rho_{\epsilon_2}(i,j,g,a)))) \\
&= (\mathsf{glue}(a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}(i,j,g,a)))^{-1} \cdot \mathsf{glue}(a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}(i,j,g,a)) \\
&\quad (\mathsf{ap}_{-^{-1} \cdot \mathsf{glue}(a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}(i,j,g,a))}(\rho_{\xi}(\mathsf{inl}(i,j,g,a)))) \\
&= \mathsf{refl}_{\mathsf{inr}(\mathsf{inr}(i,j,g,\mathsf{pr}_2(F_i)(a)))} \\
\mathsf{transp}^{x \mapsto \xi(\tilde{\xi}(x))=x}(\mathsf{glue}_{\bigvee \vee \bigvee}(a), \mathsf{refl}_{\mathsf{inl}(\mathsf{inl}(a))}) &= \mathsf{ap}_{\xi}(\mathsf{ap}_{\tilde{\xi}}(\mathsf{glue}(a)))^{-1} \cdot \mathsf{glue}(a) \\
&= \mathsf{refl}_{\mathsf{inl}(\mathsf{inl}(a))} \cdot \mathsf{glue}(a) \\
&\equiv \mathsf{glue}(a)
\end{aligned}$$

□

Now, define $\sigma : \left(\bigvee_{i,j,g} \mathsf{pr}_1(F_i)\right) \vee \left(\bigvee_{i,j,g} \mathsf{pr}_1(F_i)\right) \to \bigvee_i \mathsf{pr}_1(F_i)$ by double induction on pushouts through the commuting square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathsf{refl}_{\mathsf{inl}(a)}"] & \bigvee_{i,j,g} \mathsf{pr}_1(F_i) \arrow[d, "\alpha_2"] \\
\bigvee_{i,j,g} \mathsf{pr}_1(F_i) \arrow[r, "\alpha_1"'] & \bigvee_i \mathsf{pr}_1(F_i)
\end{tikzcd}$$

Here, $\alpha_1$ and $\alpha_2$ are induced by the commuting squares

$$\begin{tikzcd}
\left(\sum_{(i,j):\Gamma_0\times\Gamma_0} \Gamma_1(i,j)\right) \times A \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathsf{glue}_{\bigvee_i \mathsf{pr}_1(F_i)}(i,a)"] & \sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)} \mathsf{pr}_1(F_i) \arrow[d, "\mathsf{inr}(i,x)"] \\
A \arrow[r, "a\mapsto\mathsf{inl}(a)"'] & \bigvee_i \mathsf{pr}_1(F_i)
\end{tikzcd}$$

$$\begin{tikzcd}
\left(\sum_{(i,j):\Gamma_0\times\Gamma_0} \Gamma_1(i,j)\right) \times A \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathsf{glue}_{\bigvee_i \mathsf{pr}_1(F_i)}(j,a)\cdot\mathsf{ap}_{\mathsf{inr}(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}"] & \sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)} \mathsf{pr}_1(F_i) \arrow[d, "\mathsf{inr}(j,\mathsf{pr}_1(F_{i,j,g})(x))"] \\
A \arrow[r, "a\mapsto\mathsf{inl}(a)"'] & \bigvee_i \mathsf{pr}_1(F_i)
\end{tikzcd}$$

respectively. We have a map

$$\begin{tikzcd}
V_1 \arrow[d, equal] & V_2 \arrow[l, "\delta_1"'] \arrow[r, "\delta_2"] \arrow[d, "\xi", "\simeq"'] & V_3 \arrow[d, equal] \\
\bigvee_{i,j,g} \mathsf{pr}_1(F_i) & \left(\bigvee_{i,j,g} \mathsf{pr}_1(F_i)\right) \vee \left(\bigvee_{i,j,g} \mathsf{pr}_1(F_i)\right) \arrow[l, "\mathsf{id}\vee\mathsf{id}"] \arrow[r, "\sigma"'] & \bigvee_i \mathsf{pr}_1(F_i)
\end{tikzcd}$$

of spans. Indeed, we have homotopies

$$
\begin{aligned}
(\mathsf{id} \vee \mathsf{id})\,(\xi(\mathsf{inl}(a))) &\equiv \mathsf{inl}(a) \\
&\equiv \delta_1(\mathsf{inl}(a)) \\
(\mathsf{id} \vee \mathsf{id})\,(\xi(\mathsf{inr}(\mathsf{inl}(i,j,g,x)))) &\equiv \mathsf{inr}(i,j,g,x) \\
&\equiv \delta_1(\mathsf{inr}(\mathsf{inl}(i,j,g,x))) \\
(\mathsf{id} \vee \mathsf{id})\,(\xi(\mathsf{inr}(\mathsf{inr}(i,j,g,x)))) &\equiv \mathsf{inr}(i,j,g,x) \\
&\equiv \delta_1(\mathsf{inr}(\mathsf{inr}(i,j,g,x))) \\
\mathsf{transp}^{x \mapsto (\mathsf{id} \vee \mathsf{id})(\xi(x)) = \delta_1(x)}(\mathsf{glue}_{V_2}(\mathsf{inl}(i,j,g)), \mathsf{refl}_{\mathsf{inl}(a)}) &= \mathsf{ap}_{\mathsf{id} \vee \mathsf{id}}(\mathsf{ap}_{\xi}(\mathsf{glue}(\mathsf{inl}(i,j,g))))^{-1} \cdot \mathsf{ap}_{\delta_1}(\mathsf{glue}(\mathsf{inl}(i,j,g))) \\
&= \mathsf{ap}_{\underbrace{(\mathsf{id} \vee \mathsf{id}) \circ \mathsf{inl}}_{\mathsf{id}}}(\mathsf{glue}(i,j,g,a))^{-1} \cdot \mathsf{glue}(i,j,g,a) \\
&= \mathsf{refl}_{\mathsf{inr}(i,j,g,\mathsf{pr}_2(F_i)(a))} \\
\mathsf{transp}^{x \mapsto (\mathsf{id} \vee \mathsf{id})(\xi(x)) = \delta_1(x)}(\mathsf{glue}_{V_2}(\mathsf{inr}(i,j,g)), \mathsf{refl}_{\mathsf{inl}(a)}) &= \mathsf{ap}_{\mathsf{id} \vee \mathsf{id}}(\mathsf{ap}_{\xi}(\mathsf{glue}(\mathsf{inr}(i,j,g))))^{-1} \cdot \mathsf{ap}_{\delta_1}(\mathsf{glue}(\mathsf{inr}(i,j,g))) \\
&= \mathsf{ap}_{\underbrace{(\mathsf{id} \vee \mathsf{id}) \circ \mathsf{inr}}_{\mathsf{id}}}(\mathsf{glue}(i,j,g,a))^{-1} \cdot \mathsf{glue}(i,j,g,a) \\
&= \mathsf{refl}_{\mathsf{inr}(i,j,g,\mathsf{pr}_2(F_i)(a))} \\
\sigma(\xi(\mathsf{inl}(a))) &\equiv \mathsf{inl}(a) \\
&\equiv \delta_2(\mathsf{inl}(a)) \\
\sigma(\xi(\mathsf{inr}(\mathsf{inl}(i,j,g,x)))) &\equiv \sigma(\mathsf{inl}(\mathsf{inr}(i,j,g,x))) \\
&\equiv \alpha_1(\mathsf{inr}(i,j,g,x)) \\
&\equiv \mathsf{inr}(i,x) \\
&\equiv \delta_2(\mathsf{inr}(\mathsf{inl}(i,j,g,x))) \\
\sigma(\xi(\mathsf{inr}(\mathsf{inr}(i,j,g,x)))) &\equiv \sigma(\mathsf{inr}(\mathsf{inr}(i,j,g,x))) \\
&\equiv \alpha_2(\mathsf{inr}(i,j,g,x)) \\
&\equiv \mathsf{inr}(j, \mathsf{pr}_1(F_{i,j,g})(x)) \\
&\equiv \delta_2(\mathsf{inr}(\mathsf{inr}(i,j,g,x))) \\
\mathsf{transp}^{x \mapsto \sigma(\xi(x)) = \delta_2(x)}(\mathsf{glue}_{V_2}(\mathsf{inl}(i,j,g)), \mathsf{refl}_{\mathsf{inl}(a)}) &= \mathsf{ap}_{\sigma}(\mathsf{ap}_{\xi}(\mathsf{glue}(\mathsf{inl}(i,j,g))))^{-1} \cdot \mathsf{ap}_{\delta_2}(\mathsf{glue}(\mathsf{inl}(i,j,g))) \\
&= \mathsf{ap}_{\underbrace{\sigma \circ \mathsf{inl}}_{\alpha_1}}(\mathsf{glue}(i,j,g,a))^{-1} \cdot \mathsf{glue}(i,a) \\
&= \mathsf{glue}(i,a)^{-1} \cdot \mathsf{glue}(i,a) \\
&= \mathsf{refl}_{\mathsf{inr}(i,\mathsf{pr}_2(F_i)(a))} \\
\mathsf{transp}^{x \mapsto \sigma(\xi(x)) = \delta_2(x)}(\mathsf{glue}_{V_2}(\mathsf{inr}(i,j,g)), \mathsf{refl}_{\mathsf{inl}(a)}) &= \mathsf{ap}_{\sigma}(\mathsf{ap}_{\xi}(\mathsf{glue}(\mathsf{inr}(i,j,g))))^{-1} \cdot \mathsf{ap}_{\delta_2}(\mathsf{glue}(\mathsf{inr}(i,j,g))) \\
&= \mathsf{ap}_{\underbrace{\sigma \circ \mathsf{inr}}_{\alpha_2}}(\mathsf{glue}(i,j,g,a))^{-1} \cdot \mathsf{glue}(j,a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a))) \\
&= \left(\mathsf{glue}(j,a) \cdot \mathsf{ap}_{\mathsf{inr}(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a))\right)^{-1} \cdot \mathsf{glue}(j,a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a))) \\
&= \mathsf{refl}_{\mathsf{inr}(j,\mathsf{pr}_2(F_j)(a))}
\end{aligned}
$$

This induces an equivalence $\tau_2 : P_V \xrightarrow{\simeq} \mathsf{PW}$ of pushouts.

**Lemma 4.5.2.** *We have an equivalence*

$$\begin{tikzcd}
\mathsf{colim}_\Gamma A \arrow[r, "\psi"] \arrow[d, "w_0"', "\simeq"] & \mathsf{colim}_\Gamma(\mathcal{F}(F)) \arrow[d, "\simeq"', "w_1"] \\
H_2 \arrow[r, "\eta_1"'] & H_1
\end{tikzcd} \tag{$\star$}$$

*between $\psi$ and $\eta_1$.*

*Proof.* Define $w_0$ and $w_1$ by the following cocones under $A$ and $\mathcal{F}(F)$, respectively.

$$\begin{tikzcd}
A \arrow[rr, "\mathsf{id}_A"] \arrow[dr, "\mathsf{inr}(i,-)"'] & \arrow[d, phantom, "\mathsf{glue}_{H_2}(\mathsf{inr}(i,j,g,a))^{-1} \cdot \mathsf{glue}_{H_2}(\mathsf{inl}(i,j,g,a))"] & A \arrow[dl, "\mathsf{inr}(j,-)"] \\
& H_2 &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{pr}_1(F_i) \arrow[rr, "\mathsf{pr}_1(F_{i,j,g})"] \arrow[dr, "\mathsf{inr}(i,-)"'] & \arrow[d, phantom, "\mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x))"] & \mathsf{pr}_1(F_j) \arrow[dl, "\mathsf{inr}(j,-)"] \\
& H_1 &
\end{tikzcd}$$

To see that the square ($\star$) commutes, proceed by induction on $\mathsf{colim}_\Gamma A$. Note that

$$\begin{aligned}
&\eta_1(w_0(\iota_i(a))) \\
\equiv\ &\eta_1(\mathsf{inr}(i,a)) \\
\equiv\ &\mathsf{inr}(i, \mathsf{pr}_2(F_i)(a)) \\
\equiv\ &w_1(\iota_i(\mathsf{pr}_2(F_i)(a))) \\
\equiv\ &w_1(\psi(\iota_i(a)))
\end{aligned}$$

and

$$
\begin{array}{c}
\mathsf{transp}^{x\mapsto \eta_1(w_0(x))=w_1(\psi(x))}(\kappa_{i,j,g}(a), \mathsf{refl}_{\mathsf{inr}(j,\mathsf{pr}_2(F_j)(a))}) \\
\| \\
\mathsf{ap}_{\eta_1}(\mathsf{ap}_{w_0}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{ap}_{w_1}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a))) \\
\| \\
\mathsf{ap}_{\eta_1}(\mathsf{glue}(\mathsf{inl}(i,j,g,a)))^{-1} \cdot \mathsf{ap}_{\eta_1}(\mathsf{glue}(\mathsf{inr}(i,j,g,a))) \cdot \mathsf{ap}_{w_1\circ\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{ap}_{w_1}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
\| \\
\mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,\mathsf{pr}_2(F_i)(a))) \cdot \mathsf{ap}_{\mathsf{inr}(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{inr}(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,\mathsf{pr}_2(F_i)(a))) \\
\| \\
\mathsf{refl}_{\mathsf{inr}(i,\mathsf{pr}_2(F_i)(a))}
\end{array}
$$

Next, define quasi-inverses $y_0$ and $y_1$ of $w_0$ and $w_1$, respectively, by recursion on puhsouts:

$$
\begin{tikzcd}
\left(\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right)\times A\right)+\left(\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right)\times A\right) \arrow[rr] \arrow[d] & & \Gamma_0\times A \arrow[ddr, bend left, "(i,a)\mapsto\iota_i(\mathsf{pr}_2(F_i)(a))"] & \\
\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right)\times A \arrow[rr, phantom, "\kappa_{i,j,g}(a)\ +\ \mathsf{refl}_{\iota_j(\mathsf{pr}_2(F_j)(a))}"] \arrow[drrr, bend right, "(i,j,g,a)\mapsto\iota_j(\mathsf{pr}_2(F_j)(a))"'] & & H_2 \arrow[dr, dashed, "y_0"] & \\
& & & \mathsf{colim}_\Gamma A
\end{tikzcd}
$$

$$
\begin{tikzcd}
\left(\sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)}\mathsf{pr}_1(F_i)\right)+\left(\sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)}\mathsf{pr}_1(F_i)\right) \arrow[rr] \arrow[d] & & \sum_{i:\Gamma_0}\mathsf{pr}_1(F_i) \arrow[ddr, bend left, "(i,x)\mapsto\iota_i(x)"] & \\
\sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)}\mathsf{pr}_1(F_i) \arrow[rr, phantom, "\kappa_{i,j,g}(x)\ +\ \mathsf{refl}_{\iota_j(F_{i,j,g}(x))}"] \arrow[drrr, bend right, "(i,j,g,x)\mapsto\iota_j(\mathsf{pr}_1(F_{i,j,g})(x))"'] & & H_1 \arrow[dr, dashed, "y_1"] & \\
& & & \mathsf{colim}_\Gamma(\mathcal{F}(F))
\end{tikzcd}
$$

On the one hand,

$$
\begin{aligned}
y_1(w_1(\iota_i(x))) &\equiv y_1(\mathsf{inr}(i,x)) \equiv \iota_i(x) \\
\mathsf{transp}^{z\mapsto y_1(w_1(z))=z}(\kappa_{i,j,g}(x), \mathsf{refl}_{\iota_j(\mathsf{pr}_1(F_{i,j,g})(x))}) &= \mathsf{ap}_{y_1}(\mathsf{ap}_{w_1}(\kappa_{i,j,g}(x)))^{-1} \cdot \kappa_{i,j,g}(x) \\
&= \mathsf{ap}_{y_1}(\mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x)))^{-1} \cdot \kappa_{i,j,g}(x) \\
&= \kappa_{i,j,g}(x)^{-1} \cdot \kappa_{i,j,g}(x) \\
&= \mathsf{refl}_{\iota_i(x)}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
w_1(y_1(\mathsf{inl}(i,j,g,x))) &\equiv w_1(\iota_j(\mathsf{pr}_1(F_{i,j,g})(x))) \\
&\equiv \mathsf{inr}(j, \mathsf{pr}_1(F_{i,j,g})(x)) \\
&= \mathsf{inl}(i,j,g,x) && (\mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1}) \\
w_1(y_1(\mathsf{inr}(i,x))) &\equiv w_1(\iota_i(x)) \equiv \mathsf{inr}(i,x) \\
\mathsf{transp}^{z \mapsto w_1(y_1(z))=z}(\mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x)), \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1}) &= \mathsf{ap}_{w_1}(\mathsf{ap}_{y_1}(\mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x))))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x)) \\
&= \mathsf{ap}_{w_1}(\kappa_{i,j,g}(x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x)) \\
&= \left(\mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x))\right)^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x)) \\
&= \mathsf{refl}_{\mathsf{inr}(i,x)} \\
\mathsf{transp}^{z \mapsto w_1(y_1(z))=z}(\mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x)), \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1}) &= \mathsf{ap}_{w_1}(\mathsf{ap}_{y_1}(\mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x)) \\
&= \mathsf{ap}_{w_1}(\mathsf{refl}_{\iota_j(\mathsf{pr}_1(F_{i,j,g})(x))})^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x)) \\
&= \mathsf{refl}_{\mathsf{inr}(j, \mathsf{pr}_1(F_{i,j,g})(x))}.
\end{aligned}
$$

Likewise, we find that $y_0 \circ w_0 \sim \mathsf{id}_{\mathsf{colim}_\Gamma A}$ and $w_0 \circ y_0 \sim \mathsf{id}_{H_2}$. □

The map ($\star$) fits into an equivalence

$$
\begin{tikzcd}
A \arrow[d, "\mathsf{inl}"', "\simeq"] \arrow[dr, phantom, "\mathsf{glue}_{H_3}(a)^{-1}"] & \mathsf{colim}_\Gamma A \arrow[l, "{[\mathsf{id}_A]}"'] \arrow[r, "\psi"] \arrow[d, "w_0"', "\simeq"] & \mathsf{colim}_\Gamma(\mathcal{F}(F)) \arrow[d, "\simeq"', "w_1"] \\
H_3 & H_2 \arrow[l, "\eta_2"] \arrow[r, "\eta_1"'] & H_1
\end{tikzcd}
$$

of spans, which gives us an equivalence $\tau_0 : \mathsf{colim}_\Gamma^A F \xrightarrow{\simeq} P_H$ of pushouts.

**Corollary 4.5.3.** *We have an equivalence* $T_F : \mathsf{colim}_\Gamma^A F \xrightarrow{\simeq} \mathsf{PW}$ *such that*

$$
\begin{aligned}
T_F(\mathsf{inl}(a)) &\equiv \mathsf{inl}(\mathsf{inl}(a)) \\
T_F(\mathsf{inr}(\iota_i(x))) &\equiv \mathsf{inr}(\mathsf{inr}(i,x)).
\end{aligned}
$$

*Proof.* Take $T_F :\equiv \tau_2 \circ \tau_1 \circ \tau_0$. □

# 5 Universality of colimits

Let $A : \mathcal{U}$. Let $\Gamma$ be a graph and $F$ be an $A$-diagram over $\Gamma$. We say that $\mathsf{colim}^A_\Gamma(F)$ is *universal*, or *pullback-stable*, if for every pullback square

$$\begin{tikzcd} \mathsf{colim}^A_\Gamma(F) \times_V Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "h"] \\ \mathsf{colim}^A_\Gamma(F) \arrow[r, "f"'] & V \end{tikzcd} \tag{pb}$$

in $A/\mathcal{U}$, the canonical map

$$\sigma_{f,h} \;\;:\;\; \mathsf{colim}^A_{i:\Gamma}(F_i \times_V Y) \to_A \mathsf{colim}^A_\Gamma(F) \times_V Y$$

is an equivalence.

**Lemma 5.0.1.** *The forgetful functor $\mathcal{F} : A/\mathcal{U} \to A$ preserves limits.*

*Proof.* The functor $\mathcal{F}$ is right adjoint to the functor $X \mapsto X + A$, so it preserves limits. $\square$

**Theorem 5.0.2.** *All colimits in $\mathcal{U}$ are universal.*

We have formalized Theorem 5.0.2 in Agda.

**Corollary 5.0.3.** *For each tree $\Gamma$ and each $A$-diagram $F$ over $\Gamma$, the colimit $\mathsf{colim}^A_\Gamma(F)$ is universal.*

*Proof.* Suppose that $\Gamma$ and consider the pullback square (pb). By Corollary 4.4.6 combined with Theorem 5.0.2, the function

$$\mathsf{pr}_1(\mathsf{colim}^A_{i:\Gamma}(F_i \times_V Y)) \xrightarrow{\mathsf{pr}_1(\sigma_{f,h})} \mathsf{pr}_1(\mathsf{colim}^A_\Gamma(F)) \times_{\mathsf{pr}_1(V)} \mathsf{pr}_1(Y)$$

is an equivalence. The codomain is in this form because $\mathcal{F}$ preserves pullbacks by Lemma 5.0.1. It follows that $\sigma_{f,h}$ is an equivalence. $\square$

**Note 5.0.4.** We can construct pullbacks in $A/\mathcal{U}$ as follows. Consider a cospan $S$

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

in $A/\mathcal{U}$ and form the standard pullback

$$\mathsf{pr}_1(X) \times_{\mathsf{pr}_1(Z)} \mathsf{pr}_1(Y) \;\coloneqq\; \sum_{x:\mathsf{pr}_1(X)} \sum_{y:\mathsf{pr}_1(Y)} \mathsf{pr}_1(f)(x) = \mathsf{pr}_1(g)(y)$$

of $\mathcal{F}(S)$ in $\mathcal{U}$ [2, Definition 4.1.1]. Define $\mu_{f,g} : A \to \mathsf{pr}_1(X) \times_{\mathsf{pr}_1(Z)} \mathsf{pr}_1(Y)$ by

$$a \;\mapsto\; \big(\mathsf{pr}_2(X)(a), \mathsf{pr}_2(Y)(a), \mathsf{pr}_2(f)(a) \cdot \mathsf{pr}_2(g)(a)^{-1}\big)$$

Now we have a cone

$$
\begin{tikzcd}[column sep=large]
\overbrace{\left(\mathsf{pr}_1(X) \times_{\mathsf{pr}_1(Z)} \mathsf{pr}_1(Y), \mu_{f,g}\right)}^{\Phi} \arrow[r, "{(\pi_y, \mathsf{refl}_x)}"] \arrow[d, "{(\pi_x, \mathsf{refl}_y)}"'] \arrow[dr, phantom, "{\langle (x,y,p) \mapsto p, H_p \rangle}"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}
\tag{sq}
$$

over $S$ where $H_p(a)$ denotes the evident path $\left(\mathsf{pr}_2(f)(a) \cdot \mathsf{pr}_2(g)(a)^{-1}\right)^{-1} \cdot \mathsf{pr}_2(f)(a) = \mathsf{pr}_2(g)(a)$ for each $a : A$. We have used $\pi$ to denote field projection for a $\Sigma$-type. We claim that (sq) is a pullback square, i.e., the function

$$(\mathsf{sq} \circ -) \;:\; ((T, f_T) \to_A \Phi) \to \mathsf{Cone}((T, f_T)\,; f, g)$$

is an equivalence for each $(T, f_T) : A/\mathcal{U}$. Indeed, for all $K :\equiv (k_1, k_2, \langle q, Q \rangle) : \mathsf{Cone}((T, f_T)\,; f, g)$, the fiber $\mathsf{fib}_{(\mathsf{sq} \circ -)}(K)$ is equivalent to the type of data

$$
\begin{aligned}
d &: T \to \mathsf{pr}_1(X) \times_{\mathsf{pr}_1(Z)} \mathsf{pr}_1(Y) \\
h_1 &: \pi_x \circ d \sim \mathsf{pr}_1(k_1) \\
h_2 &: \pi_y \circ d \sim \mathsf{pr}_1(k_2) \\
\tau &: \prod_{t:T} \mathsf{ap}_{\mathsf{pr}_1(f)}(h_1(t)) \cdot q(t) \cdot \mathsf{ap}_{\mathsf{pr}_1(g)}(h_2(t))^{-1} = \pi_p(d(t))
\end{aligned}
\qquad
\begin{aligned}
d_p &: d \circ f_T \sim \mu_{f,g} \\
H_1 &: \prod_{a:A} h_1(f_T(a))^{-1} \cdot \mathsf{ap}_{\pi_x}(d_p(a)) = \mathsf{pr}_2(k_1)(a) \\
H_2 &: \prod_{a:A} h_2(f_T(a))^{-1} \cdot \mathsf{ap}_{\pi_y}(d_p(a)) = \mathsf{pr}_2(k_2)(a) \\
\nu &: \prod_{a:A} \Theta(\tau, d_p, H_1, H_2, a) = Q(a)
\end{aligned}
$$

where $\Theta(\tau, d_p, H_1, H_2, a)$ denotes the chain of paths

$$
\begin{array}{c}
q(f_T(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a) \\
\Big\Vert \, \scriptstyle{via\ \tau(f_T(a))} \\
\mathsf{ap}_{\mathsf{pr}_1(g)}(h_2(f_T(a)))^{-1} \cdot \pi_p(d(f_T(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(h_1(f_T(a))) \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a) \\
\Big\Vert \, \scriptstyle{via\ H_1(a)\ and\ H_2(a)} \\
\mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a) \cdot \mathsf{ap}_{\pi_y}(d_p(a))^{-1}) \cdot \pi_p(d(f_T(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{ap}_{\pi_x}(d_p(a)) \cdot \mathsf{pr}_2(k_1)(a)^{-1}) \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a) \\
\Big\Vert \\
\mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{ap}_{\pi_y}(d_p(a))^{-1}) \cdot \pi_p(d(f_T(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{ap}_{\pi_x}(d_p(a))) \cdot \mathsf{pr}_2(f)(a) \\
\Big\Vert \, \scriptstyle{via\ d_p(a)} \\
\mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{ap}_{\pi_y}(d_p(a))^{-1}) \cdot \mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{ap}_{\pi_y}(d_p(a))) \cdot \left(\mathsf{pr}_2(f)(a) \cdot \mathsf{pr}_2(g)(a)^{-1}\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{ap}_{\pi_x}(d_p(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{ap}_{\pi_x}(d_p(a))) \cdot \mathsf{pr}_2(f)(a) \\
\Big\Vert \\
\mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a)) \cdot \mathsf{pr}_2(g)(a)
\end{array}
$$

We can contract the four left-hand fields to the point defined by

$$
\begin{aligned}
d(t) &:\equiv (\mathsf{pr}_1(k_1)(t), \mathsf{pr}_1(k_2)(t), q(t)) \\
h_1(t) &:\equiv \mathsf{refl}_{\mathsf{pr}_1(k_1)(t)} \\
h_2(t) &:\equiv \mathsf{refl}_{\mathsf{pr}_1(k_2)(t)} \\
\tau(t) &:\equiv \mathsf{RId}(q(t))
\end{aligned}
$$

because $\mathsf{pr}_1(X) \times_{\mathsf{pr}_1(Z)} \mathsf{pr}_1(Y)$ is the standard pullback of $\mathcal{F}(S)$. Therefore, it suffices to prove that the type of data

$$
\begin{aligned}
d_1 &: \mathsf{pr}_1(k_1) \circ f_T \sim \mathsf{pr}_2(X) \\
H_1 &: d_1 \sim \mathsf{pr}_2(k_1) \\
d_2 &: \mathsf{pr}_2(k_2) \circ f_T \sim \mathsf{pr}_2(Y) \\
H_2 &: d_2 \sim \mathsf{pr}_2(k_2) \\
d_3 &: \prod_{a:A} q(f_T(a)) = \mathsf{ap}_{\mathsf{pr}_1(f)}(d_1(a)) \cdot \mathsf{pr}_2(f)(a) \cdot \mathsf{pr}_2(g)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(g)}(d_2(a))^{-1} \\
\nu &: \prod_{a:A} \Theta(\tau, d_3, H_1, H_2, a) = Q(a)
\end{aligned}
$$

is contractible. We can contract the first four fields to $(\mathsf{pr}_2(k_1), \mathsf{refl}, \mathsf{pr}_2(k_2), \mathsf{refl})$. This leaves us with the type of data

$$
\begin{aligned}
\upsilon_1 &: \prod_{a:A} q(f_T(a))^{-1} = \mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a)) \cdot \mathsf{pr}_2(g)(a) \cdot \mathsf{pr}_2(f)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a))^{-1} \\
\upsilon_2 &: \prod_{a:A} \Psi(\upsilon_1, a) = Q(a)
\end{aligned}
$$

where $\Psi(\upsilon_1, a)$ denotes the chain

$$
\begin{gathered}
q(f_T(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a) \\
\Big\| \, \mathsf{ap}_{- \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a)}(\nu_1) \\
\left(\mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a)) \cdot \mathsf{pr}_2(g)(a) \cdot \mathsf{pr}_2(f)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a))^{-1}\right) \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a) \\
\Big\| \\
\mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a)) \cdot \mathsf{pr}_2(g)(a)
\end{gathered}
$$

As the function $- \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a)$ is an equivalence, it follows that this type is contractible, as desired.

*Remark.* The category $A/\mathcal{U}$ is usually *not* LCC. Indeed, it is not LCC whenever $A$ is connected. In this case, suppose, for example, that $\Gamma$ is the discrete graph on $\mathbf{2}$ and define the $A$-diagram $F$ over $\Gamma$

by $F_i :\equiv (A, \mathsf{id}_A)$ for each $i : \mathbf{2}$. Then

$$\mathsf{pr}_1(\mathsf{colim}^A_\Gamma(F) \times_{\mathbf{1}} \mathbf{2}) \simeq A + A \not\simeq A + A + A \simeq \mathsf{pr}_1(\mathsf{colim}^A_{i:\Gamma}(F_i \times_{\mathbf{1}} \mathbf{2}))$$

where $A \to \mathbf{2}$ is defined by, say, $a \mapsto 0$.

By the classical adjoint functor theorem, a locally presentable $\infty$-category is LCC if and only if all its colimits are universal. In this light, Corollary 5.0.3 may be seen as a lower bound on how close $A/\mathcal{U}$ is to being LCC.

# 6 Preservation of connected maps

Let $\Gamma$ be a graph. Consider the wild category $\mathcal{D}_\Gamma$ of all diagrams over $\Gamma$ valued in $\mathcal{U}$. Its object type is $\sum_{F:\Gamma_0 \to \mathcal{U}} \prod_{i,j:\Gamma_0} \Gamma_1(i,j) \to F_i \to F_j$, and the type of morphisms from $F$ to $G$ is $F \Rightarrow G$. The identity transformation is

$$\mathsf{id}_F \;:\equiv\; \left(\lambda i.\, \mathsf{id}_{F_i}, \lambda i \lambda j \lambda g \lambda x. \mathsf{refl}_{F_{i,j,g}(x)}\right)$$

and the composition of natural transformations is the function

$$\begin{aligned}
\circ \;:\; & (G \Rightarrow H) \to (F \Rightarrow G) \to (F \Rightarrow H) \\
(\beta, q) \circ (\alpha, p) \;:\equiv\; & \left(\lambda i. \beta_i \circ \alpha_i, \lambda i \lambda j \lambda g \lambda x. \underbrace{q_{i,j,g}(\alpha(x)) \cdot \mathsf{ap}_{\beta_j}(p_{i,j,g}(x))}_{(q*p)(i,j,g,x)}\right).
\end{aligned}$$

**Lemma 6.0.1.** *Consider a family of types*

$$P \;:\; \prod_{\alpha,\beta:\prod_{i:\Gamma_0} F_i \to G_i} \prod_{p,q:\prod_{i,j,g} G_{i,j,g} \circ \alpha_i \sim \alpha_j \circ F_{i,j,g}} \sum_{W:\prod_{i:\Gamma_0} \alpha_i \sim \beta_i} \left(\prod_{i,j,g} \prod_{x:F_i} p_{i,j,g}(x) = \mathsf{ap}_{G_{i,j,g}}(W_i(x)) \cdot q_{i,j,g}(x) \cdot W_j(F_{i,j,g}(x))^{-1}\right) \to \mathcal{U}$$

*Suppose that for each $(\alpha, p) : F \Rightarrow G$, we have a term*

$$c(\alpha, p) \;:\; P(\alpha, \alpha, p, p, \lambda i \lambda x. \mathsf{refl}_{\alpha_i(x)}, \lambda i \lambda j \lambda g \lambda x. \mathsf{RId}(p_{i,j,g}(x))^{-1}).$$

*Then we have a function*

$$s \;:\; \prod_{\alpha,\beta:\prod_{i:\Gamma_0} F_i \to G_i} \prod_{p,q:\prod_{i,j,g} G_{i,j,g} \circ \alpha_i \sim \alpha_j \circ F_{i,j,g}} \prod_{W:\prod_{i:\Gamma_0} \alpha_i \sim \beta_i} \prod_{C:\prod_{i,j,g} \prod_{x:F_i} p_{i,j,g}(x) = \mathsf{ap}_{G_{i,j,g}}(W_i(x)) \cdot q_{i,j,g}(x) \cdot W_j(F_{i,j,g}(x))^{-1}} P(\alpha, \beta, p, q, C)$$

*along with an equality $s(\alpha, \alpha, p, p, \lambda i \lambda x. \mathsf{refl}_{\alpha_i(x)}, \lambda i \lambda j \lambda g \lambda x. \mathsf{RId}(p_{i,j,g}(x))^{-1}) = c(\alpha, p)$ for each $(\alpha, p) :$ $F \Rightarrow G$.*

*Proof.* For all $(\alpha, p), (\beta, q) : F \Rightarrow G$, we can use Theorem A.0.3 to find an equivalence $\mathsf{happly}_\Gamma$

$$(\alpha, p) = (\beta, q)$$

$$\wr\!\wr$$

$$\underbrace{\sum_{W : \prod_{i:\Gamma_0} \alpha_i \sim \beta_i} \prod_{i,j,g} \prod_{x:F_i} p_{i,j,g}(x) = \mathsf{ap}_{G_{i,j,g}}(W_i(x)) \cdot q_{i,j,g}(x) \cdot W_j(F_{i,j,g}(x))^{-1}}_{(\alpha,p) \sim_\Gamma (\beta,q)}$$

Thus, the family

$$(\beta, q) \;\mapsto\; (\alpha, p) \sim_\Gamma (\beta, q)$$

pointed by $\left(\mathsf{refl}_{\alpha_i(x)}, \mathsf{RId}(p_{i,j,g}(x))^{-1}\right)$ is an identity system on $(F \Rightarrow G, (\alpha, p))$. Now Theorem A.0.2 gives us our desired section. ☐

*Notation.*

- Define $\langle W, C \rangle :\equiv \mathsf{happly}_\Gamma^{-1}(W, C)$.
- For each $(\alpha, \alpha_p) : F \Rightarrow G$, define $\alpha^* :\equiv (\alpha, \alpha_p)$.

The following three propositions are easy to verify with Lemma 6.0.1.

**Proposition 6.0.2 (Precomposition).** *Let $\alpha^*, \beta^* : G \Rightarrow H$ and $\zeta^* : F \Rightarrow G$. For every $(W, C) : \alpha^* \sim_\Gamma \beta^*$, we have a path*

$$\mathsf{ap}_{-\circ\zeta^*}(\langle W, C \rangle) \;=\; \langle \lambda i \lambda x. W_i(\zeta_i(x)), \lambda i \lambda j \lambda g \lambda x. \tau_{W,C}(i, j, g, x) \rangle$$

*between elements of the identity type*

$$\begin{tikzcd} F_i \arrow[r, "F_{i,j,g}"] \arrow[d, "\alpha_i \circ \zeta_i"'] \arrow[dr, phantom, "(\alpha_p * \zeta_p)_{i,j,g}"] & F_j \arrow[d, "\alpha_j \circ \zeta_j"] \\ H_i \arrow[r, "H_{i,j,g}"'] & H_j \end{tikzcd} \quad = \quad \begin{tikzcd} F_i \arrow[r, "F_{i,j,g}"] \arrow[d, "\beta_i \circ \zeta_i"'] \arrow[dr, phantom, "(\beta_p * \zeta_p)_{i,j,g}"] & F_j \arrow[d, "\beta_j \circ \zeta_j"] \\ H_i \arrow[r, "H_{i,j,g}"'] & H_j \end{tikzcd}$$

*where* $\tau_{W,C}(i,j,g,x)$ *denote the chain of paths*

$$\begin{array}{c}
\alpha_p(i,j,g,\zeta_i(x)) \cdot \mathsf{ap}_{\alpha_j}(\zeta_p(i,j,g,x)) \\
\| \\
{\scriptstyle \textit{via } C_{i,j,g}(\zeta_i(x))} \\
\| \\
\Big(\mathsf{ap}_{H_{i,j,g}}(W_i(\zeta_i(x))) \cdot \beta_p(i,j,g,\zeta_i(x)) \cdot W_j(G_{i,j,g}(\zeta_i(x)))^{-1}\Big) \cdot \mathsf{ap}_{\alpha_j}(\zeta_p(i,j,g,x)) \\
\| \\
{\scriptstyle \textit{homotopy naturality of } W_j \textit{ at } \zeta_p(i,j,g,x)} \\
\| \\
\Big(\mathsf{ap}_{H_{i,j,g}}(W_i(\zeta_i(x))) \cdot \beta_p(i,j,g,\zeta_i(x)) \cdot \mathsf{ap}_{\beta_j}(\zeta_p(i,j,g,x)) \cdot W_j(\zeta_j(F_{i,j,g}(x)))^{-1} \cdot \mathsf{ap}_{\alpha_j}(\zeta_p(i,j,g,x))^{-1}\Big) \cdot \mathsf{ap}_{\alpha_j}(\zeta_p(i,j,g,x)) \\
\Big\| \\
\mathsf{ap}_{H_{i,j,g}}(W_i(\zeta_i(x))) \cdot \Big(\beta_p(i,j,g,\zeta_i(x)) \cdot \mathsf{ap}_{\beta_j}(\zeta_p(i,j,g,x))\Big) \cdot W_j(\zeta_j(F_{i,j,g}(x)))^{-1}
\end{array}$$

**Proposition 6.0.3 (Postcomposition).** *Let* $\zeta^* : G \Rightarrow H$. *Let* $\alpha^*, \beta^* : F \Rightarrow G$. *For every* $(W,C) : \alpha^* \sim_\Gamma \beta^*$, *we have a path*

$$\mathsf{ap}_{\zeta^* \circ -}(\langle W, C\rangle) \;=\; \Big\langle \lambda i \lambda x.\, \mathsf{ap}_{\zeta_i}(W_i(x)), \lambda i \lambda j \lambda g \lambda x. \tau_{W,C}(i,j,g,x)\Big\rangle$$

*between elements of the identity type*

$$\begin{tikzcd}
F_i \arrow[r, "F_{i,j,g}"] \arrow[d, "\zeta_i \circ \alpha_i"'] \arrow[dr, phantom, "(\zeta_p * \alpha_p)_{i,j,g}"] & F_j \arrow[d, "\zeta_j \circ \alpha_j"] \\
H_i \arrow[r, "H_{i,j,g}"'] & H_j
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F_i \arrow[r, "F_{i,j,g}"] \arrow[d, "\zeta_i \circ \beta_i"'] \arrow[dr, phantom, "(\zeta_p * \beta_p)_{i,j,g}"] & F_j \arrow[d, "\zeta_j \circ \beta_j"] \\
H_i \arrow[r, "H_{i,j,g}"'] & H_j
\end{tikzcd}$$

*where* $\tau_{W,C}(i,j,g,x)$ *denotes the chain of equalities*

$$\begin{array}{c}
\zeta_p(i,j,g,\alpha_i(x)) \cdot \mathsf{ap}_{\zeta_j}(\alpha_p(i,j,g,x)) \\
\| \\
{\scriptstyle \textit{via } C_{i,j,g}(x)} \\
\| \\
\zeta_p(i,j,g,\alpha_i(x)) \cdot \mathsf{ap}_{\zeta_j}(\mathsf{ap}_{G_{i,j,g}}(W_i(x)) \cdot \beta_p(i,j,g,x) \cdot W_j(F_{i,j,g}(x))^{-1}) \\
\Big\| \\
\zeta_p(i,j,g,\alpha_i(x)) \cdot \mathsf{ap}_{\zeta_j \circ G_{i,j,g}}(W_i(x)) \cdot \mathsf{ap}_{\zeta_j}(\beta_p(i,j,g,x)) \cdot \mathsf{ap}_{\zeta_j}(W_j(F_{i,j,g}(x)))^{-1} \\
\| \\
{\scriptstyle \textit{homotopy naturality of } \zeta_p \textit{ at } W_i(x)} \\
\| \\
\zeta_p(i,j,g,\alpha_i(x)) \cdot \Big(\zeta_p(i,j,g,\alpha_i(x))^{-1} \cdot \mathsf{ap}_{H_{i,j,g}}(\mathsf{ap}_{\zeta_i}(W_i(x))) \cdot \zeta_p(i,j,g,\beta_i(x))\Big) \cdot \mathsf{ap}_{\zeta_j}(\beta_p(i,j,g,x)) \cdot \mathsf{ap}_{\zeta_j}(W_j(F_{i,j,g}(x)))^{-1} \\
\Big\| \\
\mathsf{ap}_{H_{i,j,g}}(\mathsf{ap}_{\zeta_i}(W_i(x))) \cdot \Big(\zeta_p(i,j,g,\beta_i(x)) \cdot \mathsf{ap}_{\zeta_j}(\beta_p(i,j,g,x))\Big) \cdot \mathsf{ap}_{\zeta_j}(W_j(F_{i,j,g}(x)))^{-1}
\end{array}$$

**Proposition 6.0.4 (Concatenation).** *Let $\alpha^*, \beta^*, \epsilon^* : F \Rightarrow G$. Let $(W, C) : \alpha^* \sim_\Gamma \beta^*$ and $(Y, D) : \beta^* \sim_\Gamma \epsilon^*$. We have a path*

$$\langle W, C\rangle \cdot \langle Y, D\rangle \ =_{\alpha^*=\epsilon^*} \ \langle \lambda i \lambda x. W_i(x) \cdot Y_i(x), \lambda i \lambda j \lambda g \lambda x. \tau_{C,D}(i, j, g, x)\rangle$$

*where $\tau_{C,D}(i, j, g, x)$ denotes the chain of equalities*

$$\begin{gathered}
\alpha_p(i,j,g,x) \\
\Big\| {\scriptstyle C_{i,j,g}(x)} \\
\mathsf{ap}_{G_{i,j,g}}(W_i(x)) \cdot \beta_p(i,j,g,x) \cdot W_j(F_{i,j,g}(x))^{-1} \\
\Big\| {\scriptstyle via\ D_{i,j,g}(x)} \\
\mathsf{ap}_{G_{i,j,g}}(W_i(x)) \cdot \Big(\mathsf{ap}_{G_{i,j,g}}(Y_i(x)) \cdot \epsilon_p(i,j,g,x) \cdot Y_j(F_{i,j,g}(x))^{-1}\Big) \cdot W_j(F_{i,j,g}(x))^{-1} \\
\Big\| \\
\mathsf{ap}_{G_{i,j,g}}(W_i(x) \cdot Y_i(x)) \cdot \epsilon_p(i,j,g,x) \cdot (W_j(F_{i,j,g}(x)) \cdot Y_j(F_{i,j,g}(x)))^{-1}
\end{gathered}$$

**Lemma 6.0.5.** *The category $\mathcal{D}_\Gamma$ has the structure of a bicategory.*

*Proof.* By the three preceding coherence laws for $\mathcal{D}_\Gamma$ together with a bunch of routine path algebra. □

**Proposition 6.0.6.** *The category $\mathcal{D}_\Gamma$ is a univalent bicategory.*

## Preservation theorem

Let $F$ and $G$ be $\mathcal{U}$-valued diagrams over $\Gamma$ and suppose that $(h, \alpha) : F \Rightarrow G$. For each $i : \Gamma_0$, we have a factorization $(\mathrm{im}_{\mathcal{L},\mathcal{R}}(h_i), s_i, t_i, p_i, \ldots)$ of $h_i$. Therefore, we have a commuting square

$$\begin{tikzcd}
F_i \arrow[r, "s_j \circ F_{i,j,g}"] \arrow[d, "s_i"'] \arrow[dr, phantom, "\mathsf{ap}_{G_{i,j,g}}(p_i(x)) \cdot \alpha_{i,j,g}(x) \cdot p_j(F_{i,j,g}(x))^{-1}"] & \mathrm{im}_{\mathcal{L},\mathcal{R}}(h_j) \arrow[d, "t_j"] \\
\mathrm{im}_{\mathcal{L},\mathcal{R}}(h_i) \arrow[r, "G_{i,j,g} \circ t_i"'] & G_j
\end{tikzcd}$$

and thus a diagonal filler

$$\begin{tikzcd}
F_i \arrow[r, "s_j \circ F_{i,j,g}"] \arrow[d, "s_i"'] \arrow[r, phantom, shift right=3, "H_{i,j,g}"] & \mathrm{im}_{\mathcal{L},\mathcal{R}}(h_j) \arrow[d, "t_j"] \\
\mathrm{im}_{\mathcal{L},\mathcal{R}}(h_i) \arrow[r, "G_{i,j,g} \circ t_i"'] \arrow[ur, dashed, "d_{i,j,g}"] \arrow[r, phantom, shift left=3, "L_{i,j,g}"] & G_j
\end{tikzcd}$$

where

$$\mathsf{ap}_{G_{i,j,g}}(p_i(x)) \cdot \alpha_{i,j,g}(x) \cdot p_j(F_{i,j,g}(x))^{-1} \ = \ L_{i,j,g}(s_i(x))^{-1} \cdot \mathsf{ap}_{t_j}(H_{i,j,g}(x))$$

for all $x : F_i$.

**Theorem 6.0.7 (Unique factorization).** *For each $F, G : \mathcal{D}_\Gamma$ and $H : F \Rightarrow G$, define the predicates*

$$\begin{aligned} \widehat{\mathcal{L}}(H) &:\equiv \prod_{i:\Gamma_0} \mathcal{L}(H_i) \\ \widehat{\mathcal{R}}(H) &:\equiv \prod_{i:\Gamma_0} \mathcal{R}(H_i). \end{aligned}$$

*Let $(h, \alpha) : F \Rightarrow G$. The type*

$$\mathsf{fact}_{\widehat{\mathcal{L}},\widehat{\mathcal{R}}}(h, \alpha) :\equiv \sum_{A:\mathcal{D}_\Gamma} \sum_{S:F\Rightarrow A} \sum_{T:A\Rightarrow G} (T \circ S \sim (h, \alpha)) \times \widehat{\mathcal{L}}(S) \times \widehat{\mathcal{R}}(T).$$

*is contractible.*

*Proof.* We have already exhibited an element of $\mathsf{fact}_{\widehat{\mathcal{L}},\widehat{\mathcal{R}}}(h, \alpha)$. Thus, it remains to prove that it's a mere proposition.

Note that $\mathsf{fact}_{\widehat{\mathcal{L}},\widehat{\mathcal{R}}}(h, \alpha)$ is equivalent to the type of data

$$\begin{aligned} A_0 &: \Gamma_0 \to \mathcal{U} \\ S_0 &: \prod_{i:\Gamma_0} F_i \to A_i \\ T_0 &: \prod_{i:\Gamma_0} A_i \to G_i \\ P &: \prod_{i:\Gamma_0} T_i \circ S_i \sim h_i \\ L &: \prod_{i:\Gamma_0} \mathcal{L}(S_i) \\ R &: \prod_{i:\Gamma_0} \mathcal{R}(T_i) \end{aligned}$$

$$\begin{aligned} A_1 &: \Gamma_0 \to \Gamma_0 \to \mathcal{U} \\ S_1 &: \prod_{i,j,g} A_{i,j,g} \circ S_i \sim S_j \circ F_{i,j,g} \\ T_1 &: \prod_{i,j,g} G_{i,j,g} \circ T_i \sim T_j \circ A_{i,j,g} \\ p &: \prod_{i,j,g} \prod_{x:F_i} (T_1 * S_1)\,(i, j, g, x) = \mathsf{ap}_{G_{i,j,g}}(P_i(x)) \cdot \alpha_{i,j,g}(x) \cdot P_j(F_{i,j,g}(x))^{-1} \end{aligned}$$

We can contract the six left-hand fields because $\mathsf{fact}_{\mathcal{L},\mathcal{R}}(h_i)$ is contractible for each $i : \Gamma_0$. Let $(A_0, S, T, P, L, R)$ be a tuple of the first six fields and call the type of the last four fields $\mathsf{coher}_{\mathcal{L},\mathcal{R}}(A_0, S, T, P, L, R)$. Let $(A_1, s, t, p)$ and $(A_1', s', t', p')$ be elements of $\mathsf{coher}_{\mathcal{L},\mathcal{R}}(A_0, S, T, P, L, R)$. We must prove that they are equal.

Note that

$$\begin{aligned} p_{i,j,g}(x) &: t_{i,j,g}(S_i(x)) \cdot \mathsf{ap}_{T_j}(s_{i,j,g}(x)) = \mathsf{ap}_{G_{i,j,g}}(P_i(x)) \cdot \alpha_{i,j,g}(x) \cdot P_j(F_{i,j,g}(x))^{-1} \\ p'_{i,j,g}(x) &: t'_{i,j,g}(S_i(x)) \cdot \mathsf{ap}_{T_j}(s'_{i,j,g}(x)) = \mathsf{ap}_{G_{i,j,g}}(P_i(x)) \cdot \alpha_{i,j,g}(x) \cdot P_j(F_{i,j,g}(x))^{-1} \end{aligned}$$

Therefore, we have two commuting squares

$$
\begin{tikzcd}
F_i \arrow[r, "S_j \circ F_{i,j,g}"] \arrow[d, "S_i"'] \arrow[dr, phantom, "s_{i,j,g}", near start] & A_j \arrow[d, "T_j"] \\
A_i \arrow[ur, "A_{i,j,g}"] \arrow[r, "G_{i,j,g} \circ T_i"'] \arrow[ur, phantom, "t_{i,j,g}^{-1}", near end, shift right=3] & G_j
\end{tikzcd}
\qquad
\begin{tikzcd}
F_i \arrow[r, "S_j \circ F_{i,j,g}"] \arrow[d, "S_i"'] \arrow[dr, phantom, "s'_{i,j,g}", near start] & A_j \arrow[d, "T_j"] \\
A_i \arrow[ur, "A'_{i,j,g}"] \arrow[r, "G_{i,j,g} \circ T_i"'] \arrow[ur, phantom, "(t'_{i,j,g})^{-1}", near end, shift right=3] & G_j
\end{tikzcd}
$$

along with two terms

$$\left(A_{i,j,g}, s_{i,j,g}, t_{i,j,g}^{-1}, p_{i,j,g}\right), \left(A'_{i,j,g}, s'_{i,j,g}, \left(t'_{i,j,g}\right)^{-1}, p'_{i,j,g}\right) \; : \; \mathsf{fill}(\lambda x.\, \mathsf{ap}_{G_{i,j,g}}(P_i(x)) \cdot \alpha_{i,j,g}(x) \cdot P_j(F_{i,j,g}(x))^{-1})$$

By Lemma 3.2.5 combined with Theorem A.0.3, this gives us data

$$
\begin{aligned}
\Delta^1_{i,j,g} \; &: \; A'_{i,j,g} \sim A_{i,j,g} \\
\Delta^2_{i,j,g}(x) \; &: \; \Delta^1_{i,j,g}(S_i(x))^{-1} \cdot s'_{i,j,g}(x) \; = \; s_{i,j,g}(x) \\
\Delta^3_{i,j,g}(x) \; &: \; t'_{i,j,g}(x) \cdot \mathsf{ap}_{T_j}(\Delta^1_{i,j,g}(x)) \; = \; t_{i,j,g}(x) \\
\Delta^4_{i,j,g}(x) \; &: \; \tau_{i,j,g}(x) \cdot \mathsf{ap}_{\left(t'_{i,j,g}(S_i(x)) \cdot \mathsf{ap}_{T_j}(\Delta^1_{i,j,g}(S_i(x)))\right) \cdot -}(\mathsf{ap}_{\mathsf{ap}_{T_j}}(\Delta^2_{i,j,g}(x))) \cdot \mathsf{ap}_{- \cdot \mathsf{ap}_{T_j}(s_{i,j,g}(x))}(\Delta^3_{i,j,g}(S_i(x))) \; = \; p'_{i,j,g}(x) \cdot p_{i,j,g}(x)^{-1},
\end{aligned}
$$

where $\tau_{i,j,g}(x)$ is the evident term of type

$$t'_{i,j,g}(S_i(x)) \cdot \mathsf{ap}_{T_j}(s'_{i,j,g}(x)) \; = \; \left(t'_{i,j,g}(S_i(x)) \cdot \mathsf{ap}_{T_j}(\Delta^1_{i,j,g}(S_i(x)))\right) \cdot \mathsf{ap}_{T_j}(\Delta^1_{i,j,g}(S_i(x))^{-1} \cdot s'_{i,j,g}(x))$$

By Theorem A.0.3 again, the type of such data is equivalent to $(A_1, s, t, p) = (A'_1, s', t', p')$, thereby completing the proof. $\square$

**Corollary 6.0.8.** *The OFS $(\mathcal{L}, \mathcal{R})$ on $\mathcal{U}$ lifts pointwise to $\mathcal{D}_\Gamma$.*

Since the functor $\mathsf{const}_\Gamma : \mathcal{U} \to \mathcal{D}_\Gamma$ clearly takes $\mathcal{R}$ to $\widehat{\mathcal{R}}$, we deduce that $\mathsf{colim}_\Gamma(-)$ takes $\widehat{\mathcal{L}}$ to $\mathcal{L}$ by Corollary 3.2.9. Now, for each $X, Y : A/\mathcal{U}$, consider the predicate $\mathcal{L}_A(f, p) :\equiv \mathcal{L}(f)$ on $X \to_A Y$. Then the functor $\mathsf{colim}^A_\Gamma$ takes $\widehat{\mathcal{L}}_A$ to $\mathcal{L}_A$. Indeed, if a map $\epsilon : \mathcal{A} \Rightarrow \mathcal{B}$ of $A$-diagrams belongs to $\widehat{\mathcal{L}}_A$, then the underlying function of $\mathsf{colim}^A_\Gamma(\epsilon)$ is precisely that induced by the morphism

$$
\begin{tikzcd}
A \arrow[d, "\mathsf{id}"'] & \mathsf{colim}_\Gamma A \arrow[l] \arrow[r] \arrow[d, "\mathsf{id}"] & \mathsf{colim}_\Gamma(\mathcal{F}(\mathcal{A})) \arrow[d, "\hat{\epsilon}"] \\
A & \mathsf{colim}_\Gamma A \arrow[l] \arrow[r] & \mathsf{colim}_\Gamma(\mathcal{F}(\mathcal{B}))
\end{tikzcd}
$$

of spans, and all three components belong to $\mathcal{L}$.

In particular, if $F$ is a *pointed* diagram over $\Gamma$ such that each $\mathsf{pr}_1(F_i)$ is $(\mathcal{L}, \mathcal{R})$-connected, then the type $\mathsf{colim}^*_\Gamma F$ is also $(\mathcal{L}, \mathcal{R})$-connected. Indeed, since $\mathsf{colim}^*_\Gamma \mathbf{1} = \mathbf{1}$, $\mathsf{colim}^*_\Gamma$ takes the unique map $F \Rightarrow_* \mathbf{1}$ of pointed diagrams to $(c, c_p) : \mathsf{colim}^*_\Gamma F \to_* \mathsf{colim}^*_\Gamma \mathbf{1}$ where $c : \mathsf{colim}^*_\Gamma F \to \mathbf{1}$ is the constant map and $c \in \mathcal{L}$.

**Example 6.0.9.**

(a) For each truncation level $n$, if each $\mathsf{pr}_1(F_i)$ is $n$-connected, then so is the underlying type of $\mathsf{colim}^*_\Gamma F$. In fact, if $F$ is an $A$-diagram with each $\mathsf{pr}_1(F_i)$ $n$-connected and $A$ is $n$-connected, then Corollary 4.5.3 shows that the underlying type of $\mathsf{colim}^A_\Gamma F$ is also $n$-connected.

(b) Let $\Gamma$ be the graph with a single point $*$ and a single edge from $*$ to itself. Define the diagram $F$ over $\Gamma$ by $F_0(*) :\equiv \mathbf{1}$ and $F_{*,*,*} :\equiv \mathsf{id}_{\mathbf{1}}$. Then $\mathsf{colim}_\Gamma(F) = S^1$, which proves that $\mathsf{colim}_\Gamma$ does not preserve $n$-connectedness when $n \geq 1$, unlike $\mathsf{colim}^*_\Gamma$.

## 6.1 Cocompleteness of $(n, k)\,\mathsf{GType}$

Let $-2 \leq n \leq \infty$ and $-1 \leq k < \infty$ be truncation levels. Recall from [4] the category $(n, k)\,\mathsf{GType}$ of *$k$-tuply groupal $n$-groupoids*. This is equivalent to the full subcategory $\mathcal{U}^*_{\geq k, \leq n+k}$ of $\mathcal{U}^*$ on $(k-1)$-connected, $(n+k)$-truncated types. We have proven that this category has all colimits over graphs.

In fact, by our second construction of $\mathsf{colim}^A_\Gamma$ (Corollary 4.5.3) combined with Corollary 4.2.5 and Lemma 3.1.9, we have that for every OFS $(\mathcal{L}, \mathcal{R})$ on $\mathcal{U}$ and every $A : \mathcal{U}$, if $A$ is $(\mathcal{L}, \mathcal{R})$-connected, then the full wild subcategory of $A/\mathcal{U}$ on $(\mathcal{L}, \mathcal{R})$-connected, $(\mathcal{L}, \mathcal{R})$-modal types has all colimits over graphs.

# 7 Weak continuity of cohomology

**Definition 7.0.1 (Finite graph).**

- We say that a type $A$ is *finite* if it's merely equivalent to a standard finite type.
- We say that a graph $\Gamma$ is *finite* if $\Gamma_0$ is finite and for each $i, j : \Gamma_0$, $\Gamma_1(i, j)$ is finite.

**Lemma 7.0.2.** *If $\Gamma$ is a finite graph, then the type $\sum_{i,j:\Gamma_0} \Gamma_1(i, j)$ is finite.*

*Proof.* This follows directly from [9, A dependent sum of finite types indexed by a finite type is finite]. $\square$

Let $\Gamma$ be a finite graph. Then every generalized cohomology theory $k : (\mathcal{U}^*)^{\mathrm{op}} \to \mathbf{Ab}$ takes pointed colimits over $\Gamma$ to weak limits in **Set**, in the sense that the universal map from the limit is epic in **Ab** (i.e., surjective). If $k$ is additive (e.g., induced by an $\Omega$-spectrum), then this holds when $\Gamma$ is a "0-choice" graph.

## Eilenberg-Steenrod cohomology theories

Let $H : \mathbb{Z} \to (\mathcal{U}^* \to \mathbf{Ab})$ be a $\mathbb{Z}$-indexed famility of functors $\mathcal{U}^* \to \mathbf{Ab}$. We say that $H$ is an *(Eilenberg-Steenrod) cohomology theory* if it satisfies the following two axioms.

- For each $n : \mathbb{Z}$, we have a natural isomorphism $H^{n+1}(\Sigma -) \xrightarrow{\sigma_n} H^n(-)$ of functors $\mathcal{U}^* \to \mathbf{Ab}$.

- For each $f : X \to_* Y$, the sequence

$$H^n(Y/X) \xrightarrow{H^n(\mathsf{cof}(f))} H^n(Y) \xrightarrow{H^n(f)} H^n(X)$$

is exact.

**Example 7.0.3.** Suppose that $E : \mathbb{Z} \to \mathcal{U}^*$ is a prespectrum, with structure maps $\epsilon_n : E_n \to_* \Omega E_{n+1}$. We have a sequence

$$\left\|X \to_* \Omega^k E_{n+k}\right\|_0 \xrightarrow{\left\|\Omega^k(\epsilon_{n+k})\circ -\right\|_0} \left\|X \to_* \Omega^{k+1} E_{n+(k+1)}\right\|_0$$

of abelian groups.[2] For each $n : \mathbb{Z}$, define

$$\begin{aligned} \widetilde{E}^n &: \mathcal{U}^* \to \mathbf{Ab} \\ \widetilde{E}^n(X) &:= \mathsf{colim}_{k:\omega} \left\|X \to_* \Omega^k E_{n+k}\right\|_0 \end{aligned}$$

where the colimit $\mathsf{colim}_{k:\omega}(G_k)$ of a sequence of abelian groups has underlying set $\mathsf{colim}_k(\mathsf{pr}_1(G_k))$ and has abelian group structure defined by induction on sequentual colimits. This is a cohomology theory. The suspension axiom is easy to verify. To see that it satisfies the exactness axiom, we first record a lemma of group theory.

**Lemma 7.0.4.** *Consider a levelwise exact sequence*

$$(A, a) \xrightarrow{(m_1, M_1)} (B, b) \xrightarrow{(m_2, M_2)} (C, c)$$

*of sequential diagrams valued in* **Ab**. *Then the sequence*

$$\mathsf{colim}_{k:\omega}(A_k) \xrightarrow{\mathsf{colim}(m_1)} \mathsf{colim}_{k:\omega}(B_k) \xrightarrow{\mathsf{colim}(m_2)} \mathsf{colim}_{k:\omega}(C_k)$$

*of abelian groups is exact.*

*Proof.* For each $k : \mathbb{N}$ and $x : A_k$,

$$\mathsf{colim}(m_2 \circ m_1)(\iota_k(x)) \;\equiv\; \iota_k(m_2(k, (m_1(k, x)))) \;=\; 0$$

because $m_2(k) \circ m_1(k) = 0$ by levelwise exactness.

Next, let $k : \mathbb{N}$ and $x : B_k$. Suppose that $\overbrace{\mathsf{colim}(m_2)(\iota_k(x))}^{\iota_k(m_2(k,x))} = 0$. We want to show that the fiber of $\mathsf{colim}(m_1)$ over $\iota_k(x)$ is nonempty. By [10, Theorem 7.4], we have an equivalence

$$(\iota_k(0) =_{\mathsf{colim}\, C} \iota_k(m_2(k, x))) \;\simeq\; \mathsf{colim}_{n:\omega}(0^{+n} =_{C_{k+n}} m_2(k, x)^{+n})$$

[2]Here we assume that addition $+ : \mathbb{Z} \times \mathbb{N} \to \mathbb{Z}$ is defined by pattern matching on the second argument.

As $\iota_k(0) = 0$, it thus suffices to prove that the fiber is nonempty under the assumption that $\mathsf{colim}_n(0^{+n} = m_2(k,x)^{+n})$. For this we proceed by induction on sequential colimits. Let $n : \mathbb{N}$ and $p : 0^{+n} = m_2(k,x)^{+n}$. By naturality of $m_2$, we see that $m_2(k,x)^{+n} = m_2(k+n, x^{+n})$. As $0^{+n} = 0$, it follows that $x^{+n}$ belongs to the kernel of $m_2(k+n)$. By levelwise exactness, this gives us an element $(d,q) : \mathsf{fib}_{m_1(k+n)}(x^{+n})$. We have that

$$\mathsf{colim}(m_1)(\iota_{k+n}(d)) \;\equiv\; \iota_{k+n}(m_1(k+n,d)) \;=\; \iota_{k+n}(x^{+n}) \;=\; \iota_k(x).$$

This proves that the fiber over $\iota_k(x)$ is nonempty. □

Now it suffices to observe that

$$\left\|{}^{Y}/_{X} \to_* \Omega^k E_{n+k}\right\|_0 \xrightarrow{\|-\circ\mathsf{cof}(f)\|_0} \left\|Y \to_* \Omega^k E_{n+k}\right\|_0 \xrightarrow{\|-\circ f\|_0} \left\|X \to_* \Omega^k E_{n+k}\right\|_0$$

is exact for every $f : X \to_* Y$ (see [5, Section 3.2.2]).

For each $n : \mathbb{Z}$, the functor $\widetilde{E}^n(-)$ computes the $-2n$-th degree $[\Sigma^\infty(-), E]_{-2n}$ of the graded hom-group in the category of prespectra [1, Proposition 2.8], where $\Sigma^\infty(X)$ denotes the suspension prespectrum of a type $X$. For example, if $E$ is the sphere spectrum, then $\widetilde{E}^{-n}(-)$ is precisely the $2n$-th homotopy group functor $\pi_{2n}(-)$ on prespectra.

If $H$ satisfies

$$H^n(S^0) \cong \mathbf{1}$$

for all $n \neq 0$, then $H$ is called *ordinary.*

If the map

$$\prod_{i:I} H^n(\mathsf{inr} \circ (i,-)) \;:\; H^n(\bigvee_{i:I} F_i) \to \prod_{i:I} H^n(F_i)$$

is an isomorphism for every set $I$ satisfying 0-choice and every family $F : I \to \mathcal{U}^*$ of pointed types, then $H$ is called *additive.*

**Example 7.0.5.** Every $\Omega$-spectrum $E : \mathbb{Z} \to \mathcal{U}^*$ induces an additive cohomology theory $\widetilde{E}$, which is ordinary if $E$ is an Eilenberg-MacLane spectrum.

## Mayer-Vietoris sequence

Suppose that $H^*$ is a cohomology theory. Consider a pushout

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & P \end{tikzcd}$$

of pointed types and pointed maps. In [5], Cavallo constructs a LES of the form

$$\cdots \longrightarrow H^{n-1}(Z) \xrightarrow{\mathsf{extglue}} H^n(P) \xrightarrow{(H^n(\mathsf{inl}),H^n(\mathsf{inr}))} H^n(X) \times H^n(Y) \xrightarrow{H^n(f)-H^n(g)} H^n(Z) \longrightarrow \cdots$$

In particular, thanks to Corollary 4.5.3 combined with Lemma 7.0.2, if $\Gamma$ is a finite graph, then we have an exact squence

$$H^n(\mathsf{colim}^*_\Gamma F) \xrightarrow{\zeta_n} \prod_{i,j,g} H^n(F_i) \times \prod_i H^n(F_i) \xrightarrow{\mu_n - \nu_n} \prod_{i,j,g} H^n(F_i) \times \prod_{i,j,g} H^n(F_i) \qquad (*)$$

for each $n : \mathbb{N}$. If $H^*$ is additive, then this holds when $\Gamma$ is just a 0-choice graph. Here, $\zeta_n$ is defined as the composite

$$\begin{tikzcd} H^n(\mathsf{colim}^*_\Gamma F) \arrow[r, dashed, "\zeta_n"] \arrow[d] & \prod_{i,j,g} H^n(F_i) \times \prod_i H^n(F_i) \\ H^n(\bigvee_{i,j,g} F_i) \times H^n(\bigvee_i F_i) \arrow[ur, "\cong \times \cong"'] & \end{tikzcd}$$

and $\mu_n$ and $\nu_n$ are defined as

$$\begin{aligned} (f, h) &\mapsto (f, \lambda i \lambda j \lambda g. H^n(F_{i,j,g})(h_j)) \\ (f, h) &\mapsto (f, \lambda i \lambda j \lambda g. h_i)\,, \end{aligned}$$

respectively. We have a cone

$$\begin{tikzcd} & H^n(\mathsf{colim}^*_\Gamma(F)) \arrow[dl, "H^n(\iota_j)"'] \arrow[dr, "H^n(\iota_i)"] & \\ H^n(F_j) \arrow[rr, "H^n(F_{i,j,g})"'] & & H^n(F_i) \end{tikzcd}$$

over $H^n(F)$ and thus a commuting diagram

$$\begin{tikzcd} H^n(\mathsf{colim}^*_\Gamma(F)) \arrow[rr, dashed, "\Delta_F"] \arrow[dr, "H^n(\iota_i)"'] & & \mathsf{lim}_\Gamma H^n(F) \arrow[dl, "\mathsf{pr}_i"] \\ & H^n(F_i) & \end{tikzcd}$$

induced by the universal property of limits in **Ab**. Thanks to the exact sequence $(*)$, we see that $\Delta_F$ is epic. Thus, $\Delta_F$ is epic in **Set** as well. Classically, this implies that $H^n(\mathsf{colim}^*_\Gamma(F))$ is a *weak limit* in **Set**. If we assume the axiom of choice inside HoTT, then we can conclude that $\Delta_F$ is *merely* a weak limit in **Set**.

# A Identity systems

In this section, we describe our main tool for characterizing path spaces of structured types: the *structure identity principle (SIP)*.

**Definition A.0.1 (Identity system).** Let $(A, a)$ be a pointed type. Consider a type family $B$ over $A$ and an element $b : B(a)$. We say that $(B, b)$ is an *identity system* on $(A, a)$ if the total space

$\sum_{x:A} B(x)$ is contractible.

**Theorem A.0.2.** *The following are logically equivalent.*

- *The family $B$ is an identity system on $(A, a)$.*
- *The family of maps $f : \prod_{x:A} (a = x) \to B(x)$ defined by $f(a, \mathsf{refl}_a) :\equiv b$ is a family of equivalences.*
- *For each family of types $P : \prod_{a:A} B(x) \to \mathcal{U}$, the function*

  $$h \mapsto h(a, b) \;:\; \left( \prod_{x:A} \prod_{y:B(x)} P(x, y) \right) \to P(a, b)$$

  *has a section, denoted by $\mathsf{ind}_P$.*

*Proof.* See [7, Theorem 11.2.2]. ☐

**Theorem A.0.3 (SIP).** *Let $(A, a)$ be a pointed type, $(B, b)$ a pointed type family over $A$, and $(C, c)$ an identity system on $(A, a)$. Consider terms*

$$\begin{aligned} D \;&:\; \prod_{x:A} B(x) \to C(x) \to \mathcal{U} \\ d \;&:\; D(a, b, c) \end{aligned}$$

*(We call $(D, d)$ a* standard notion of structure (SNS) *for $(A, a)$.) Then the following are equivalent.*

- *The total space*

  $$\sum_{y:B(a)} D(a, y, c)$$

  *is contractible.*
- *Every family of maps*

  $$((a, b) = (x, y)) \to \sum_{z:C(x)} D(x, y, z)$$

  *indexed by $(x, y) : \sum_{x:A} B(x)$ is a family of equivalences.*
- *The type family*

  $$(x, y) \;\mapsto\; \sum_{z:C(x)} D(x, y, z)$$

  *is an identity system at $(a, b) : \sum_{x:A} B(x)$.*

*Proof.* See [7, Theorem 11.6.2]. ☐

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