On Adjunctions, Colimits, and 2-groups in Homotopy Type Theory

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Outline

1. Background on homotopy type theory

2. First theme of thesis: **colimits**

   Encompassing projects on coslice colimits and wild adjunctions.

3. Second theme of thesis: **2-groups**

4. Conclusion

Dependent type theory

Martin-Löf type theory (MLTT) is a constructive formal system taking *type* and *type inhabitation* as primitive notions.

We try to prove judgments of the form

$$\Gamma \vdash t : X \qquad \textit{t has type } X \textit{ in context } \Gamma$$

We also say t *is an element / a proof of* X.

Some important type formers in MLTT:

- *the type* $X \to Y$ *of functions from type* X *to type* Y
- $(x : X) \to P(x)$ *for all* x *in* X*, we have an element of* $P(x)$*.*
- For all $x, y : X$, the *identity/path type* $x = y$, inductively generated by reflexivity $\mathsf{refl}_x : x = x$
- *type universes* $\mathcal{U}$, where $X : \mathcal{U} \iff X$ is a type

Homotopy type theory (HoTT)

HoTT is a variant of dependent type theory where

- types are interpreted as *homotopy types*: topological spaces up to continuous deformation

  interpretation in all $(\infty, 1)$*-toposes (Shulman)*

  Enables ***synthetic homotopy theory****:*
  HoTT is a domain-specific language for homotopy theory.

- proof-checking is (at least) semi-decidable (Cockx et al.), with implementations in Agda and Rocq.

  If Agda returns successfully, then we know our proof is correct.

HoTT extends MLTT with at least one of

- **the univalence axiom**

  Identity of types is the same as equivalence of types.

- **higher inductive types**

  Generalize inductive types by allowing generators of id types.

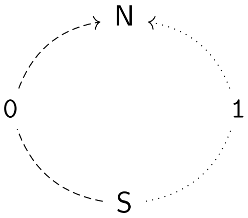
  Directly build and study higher-dimensional spaces.

Consequence: HoTT is *proof-relevant*.

- Two proofs $p_1, p_2 : x = y$ may have different properties. We need to keep track of particular proofs we construct.
- Better to think of p_1 and p_2 as possibly different paths/isomorphisms between x and y.

Example of HIT: the circle S^1

Generated by two points N, S and two paths $0, 1 : \mathsf{S} = \mathsf{N}$:



The two themes of this thesis

Central goal:

Use HITs to build and study things of interest in synthetic homotopy theory.

These things of interest come in two forms (which we call themes):

1. **colimits**

   fundamental constructions from category theory, letting us build complex spaces from simpler ones

2. **deloopings of coherent** 2**-groups**

   a way to encode 2*-dimensional groups as spaces and vice versa*

Note: extensive Agda formalization for each theme

First theme: colimits

A ***(directed) graph*** is a pair $\Gamma :\equiv (\Gamma_0, \Gamma_1)$ consisting of

- a type $\Gamma_0 : \mathcal{U}$ of vertices
- a family $\Gamma_1 : \Gamma_0 \to \Gamma_0 \to \mathcal{U}$ of edges.

$$\begin{tikzcd}
v_0 \arrow[rr, "e_2"] \arrow[d, bend right, "e_0"'] \arrow[d, bend left, "e_1"] & & v_1 \\
v_3 & & v_2 \arrow[llu, "e_3"']
\end{tikzcd}
\qquad\qquad v_i : \Gamma_0$$

Let F be a Γ-*shaped diagram* in $\mathcal{U}$, e.g.,

$$\begin{tikzcd}
F(v_0) \arrow[rr, "F(e_2)"] \arrow[d, bend right, "F(e_0)"'] \arrow[d, bend left, "F(e_1)"] & & F(v_1) \\
F(v_3) & & F(v_2) \arrow[llu, "F(e_3)"']
\end{tikzcd}
\qquad\qquad F(v_i) : \mathcal{U}$$

The ***colimit of*** F is the HIT $\mathrm{colim}_\Gamma(F)$ generated by

$$\begin{aligned}\mathrm{leg} &: (i : \Gamma_0) \to F_i \to \mathrm{colim}_\Gamma(F) \\ \mathrm{tri} &: (i, j : \Gamma_0)(g : \Gamma_1(i,j)) \to \mathrm{leg}_j \circ F_{i,j,g} \sim \mathrm{leg}_i\end{aligned}$$

$$\begin{tikzcd} F_i \arrow[rr, "F_{i,j,g}"] \arrow[dr, "\mathrm{leg}_i"'] & \arrow[d, phantom, "\mathrm{tri}_{i,j,g}"] & F_j \arrow[dl, "\mathrm{leg}_j"] \\ & \mathrm{colim}_\Gamma(F) & \end{tikzcd}$$

($f_1 \sim f_2$ means pointwise identity of functions.)

This data forms a ***cocone under*** F ***with tip*** $\mathrm{colim}_\Gamma(F)$.

Induction principle: This cocone is the categorical colimit in $\mathcal{U}$: the *initial* (roughly, smallest) cocone under F.[1]

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[1] We just consider graph-indexed colimits to avoid an infinite tower of coherences that may be undefinable in HoTT.

Adding structure to $\mathcal{U}$

Let A be a type. The ***coslice category*** $A/\mathcal{U}$ is defined by

$$\mathrm{Ob}(A/\mathcal{U}) := \sum_{X:\mathcal{U}} A \to X \qquad (\textit{elements are pairs } (X, f))$$

$$\underbrace{\mathrm{Mor}_{A/\mathcal{U}}(Y_1, Y_2)}_{Y_1 \to_A Y_2} := \sum_{\varphi : \mathrm{pr}_1(Y_1) \to \mathrm{pr}_1(Y_2)} \varphi \circ \mathrm{pr}_2(Y_1) \sim \mathrm{pr}_2(Y_2)$$

$$\begin{tikzcd}
 & A \arrow[dl, "\mathrm{pr}_2(Y_1)"'] \arrow[dr, "\mathrm{pr}_2(Y_2)"] & \\
\mathrm{pr}_1(Y_1) \arrow[rr, "\varphi"'] & & \mathrm{pr}_1(Y_2)
\end{tikzcd}$$

Coslices of the category of spaces appear often in homotopy theory, especially the category of pointed spaces, i.e., the coslice under the one-point space.

Overview of work on coslice colimits

1. Construct *coslice colimits*—colimits in $A/\mathcal{U}$—in a useful way.

   Construction should reveal their interaction with colimits in a type universe $\mathcal{U}$, *i.e., with the* colim *HIT.*

2. Use the construction to give some purely categorical results—for example, about which colimits the forgetful functor $\mathcal{F} : A/\mathcal{U} \to \mathcal{U}$ creates.

3. Use the construction to prove useful results about other areas of synthetic homotopy theory:

   - acyclic types
   - orthogonal factorization systems
   - higher group theory
   - cohomology theory

Construction of coslice colimits

1. For a diagram F in $A/\mathcal{U}$, form the following pushout in $\mathcal{U}$ (a special case of the colimit HIT):

$$\begin{tikzcd}
\mathrm{colim}_\Gamma A \arrow[r] \arrow[d] & \mathrm{colim}_\Gamma(\mathcal{F}(F)) \arrow[d] \\
A \arrow[r, "\mathrm{inl}"'] & \mathcal{P}_F \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

2. Form a cocone with tip $(\mathcal{P}_F, \mathrm{inl})$:

$$\begin{tikzcd}
F_i \arrow[rr, "F_{i,j,g}"] \arrow[dr] & & F_j \arrow[dl] \\
& (\mathcal{P}_F, \mathrm{inl}) &
\end{tikzcd}$$

3. Prove that this cocone is a colimit in $A/\mathcal{U}$.

Colimits and left adjoints

We also studied properties of colimits in ***wild categories*** (such as $A/\mathcal{U}$)—the general setting for synthetic homotopy theory.

We showed that the "standard proof" that left adjoints (a special kind of functor) preserve colimits fails for wild categories.

1. Identified the exact bicategorical (2-dimensional) coherence condition for the standard proof to go through.
2. Used it to prove that the suspension functor $\Sigma : \mathcal{U}_* \to \mathcal{U}_*$ (defined via pushouts) on pointed types preserves graph-indexed colimits.
3. Applied these results together with our construction of coslice colimits to acyclic types and higher group theory.

Extensions of colimit work

- Find an example of a left adjoint between wild categories that, internally, does not preserve colimits.

  This one was recently completed.[2]

- Prove that the join $X * - : \mathcal{U}_* \to \mathcal{U}_*$ preserves graph-indexed colimits for each pointed type X.

  This one was recently completed.[2]

- Do the same for the smash product $X \wedge - : \mathcal{U}_* \to \mathcal{U}_*$.

[2] P. Hart, On Left Adjoints Preserving Colimits in Homotopy Type Theory, arXiv:2608.28473 (2026).

Second theme: 2-groups

A 2-*group* is a 2-dimensional generalization of a group.

Goes back at least to Grothendieck in the 1970s.

We study how to classify a certain class of spaces as 2-groups.

Result: biadjoint biequivalence (a notion from higher category theory) between coherent 2-groups and pointed connected 2-types

This relationship has been considered in both HoTT and classical algebraic topology but has never been established in this form.

A critical notion

An *n-type*, or *n-truncated type*, is a type whose identity types are trivial above level n.

- A type X is a 0-type, or a *set*, if each of its loop spaces $x = x$ contains only refl_x.
- A type X is a 1-*type* if all of its loop spaces are sets.
- A type X is a 2-*type* if all of its loop spaces are 1-types.
- And so on.

Our algebraic classification of pointed connected 2-types fits into a layered description of spaces as *groupoids* (roughly, categories where all maps are invertible).

(A type X is *connected* if its set $\pi_0(X)$ of path components is trivial.)

The homotopy hypothesis (Buchholtz–van Doorn–Rijke)

Groupoids	**Spaces**
group	pointed connected 1-type
abelian group	pointed n-connected $(n+1)$-type, $n > 0$
2-group	pointed connected 2-type
braided 2-group	pointed 1-connected 3-type
symmetric 2-group	pointed n-connected $(n+2)$-type, $n > 1$
$\vdots$	$\vdots$

2-groups

A *(coherent) 2-group* (Baez and Lauda) is a 1-type G with

- a binary operation $\otimes : G \to G \to G$, called the *tensor product*
- a neutral element id
- a right unitor ρ, a left unitor λ, and an associator α for $\otimes$

$$e.g., \quad \alpha \;:\; (x, y, z : X) \to (x \otimes y) \otimes z = x \otimes (y \otimes z)$$

- a triangle identity and a pentagon identity
- an inverse operation $(-)^{-1} : G \to G$
- paths $\mathrm{linv}_x : x^{-1} \otimes x = \mathrm{id}$ and $\mathrm{rinv}_x : x \otimes x^{-1} = \mathrm{id}$ for each $x : G$ such that linv and rinv satisfy two zig-zag identities.

Example

For every pointed 2-type X, the loop space $\Omega(X)$ equipped with path composition has the structure of a 2-group.

A 2-*group morphism* $G_1 \to G_2$ is a function $f : G_1 \to G_2$ equipped with a family of paths $\mu_{x,y} : f(x) \otimes f(y) = f(x \otimes y)$ that respects the associator.

Note: This short notion of 2-group morphism is equivalent to the one that explicitly preserves all data.

Delooping a 2-group

Let G be a 2-group. We want a pointed type D with $G \cong \Omega(D)$.

Construct its *delooping* by generalizing the delooping of an ordinary group (Licata and Finster).

Form the 2-truncated HIT $\mathcal{K}_2(G)$ generated by

- a point base : $\mathcal{K}_2(G)$
- a morphism of 2-groups loop : $G \to \Omega(\mathcal{K}_2(G))$

Key feature: no path constructors for units or inverses

makes induction on $\mathcal{K}_2(G)$ much more tractable

Theorem (delooping): loop is an isomorphism.

The biequivalence

Goal: Show $\mathcal{K}_2$ and loop fit into a *bicategorical* equivalence between 2-groups and pointed connected 2-types.

Bicategories augement categories with morphisms between morphisms.

A ***(biadjoint) biequivalence*** between bicateogries $\mathcal{C}$ and $\mathcal{D}$ is a pseudofunctor[3] $F : \mathcal{C} \to \mathcal{D}$ together with

- a pseudofunctor $G : \mathcal{D} \to \mathcal{C}$
- a pseudotransformation $\varepsilon : F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$ each of whose components is an adjoint equivalence in $\mathcal{D}$
- a pseudotransformation $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$ each of whose components is an adjoint equivalence in $\mathcal{C}$
- a *single* zig-zag identity between ε and η.

[3] The prefix *pseudo* means a bicategorical extension of a 1-categorical notion.

Theorem

We have a biequivalence

$$\mathbf{2Grp} \underset{\Omega}{\overset{\mathcal{K}_2}{\rightleftarrows}} \mathbf{Ptd}^{conn}_{\leq 2}$$

These bicategories are *univalent* (Ahrens et al.): paths between objects are the same as adjoint equivalences between them.

Lemma

Let F be a pseudofunctor between univalent bicategories. The type "F is a biequivalence" is equivalent to "F is an isomorphsism (i.e., a type equivalence on both objects and morphisms)."

Corollary (via the univalence axiom)

Both $\mathcal{K}_2$ and Ω induce paths $\mathbf{2Grp} = \mathbf{Ptd}^{conn}_{\leq 2}$ of bicategories.

Extension of 2-group work

Find a tractable way to verify in HoTT the remaining two cases of the dimension-2 homotopy hypothesis—for braided 2-groups and symmetric 2-groups.

Conclusion

We have used HITs in new ways to develop two important areas of synthetic homotopy theory.

Colimits: (1) A useful construction of colimits in coslices of a type universe and (2) a study of colimits and left adjoints in general wild categories.[4]

2-groups: A biequivalence between coherent 2-groups and pointed connected 2-types[5]—an instance of the homotopy hypothesis.

Thank you!

[4] Agda code: `https://github.com/PHart3/colimits-agda`
[5] Agda code: `https://github.com/PHart3/2-groups-agda`

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