Classifying 2-Groups in Homotopy Type Theory

Perry Hart[1] and Owen Milner[2]

[1]University of Minnesota

[2]Carnegie Mellon University

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Overview

1. Background

2. Working in *homotopy type theory*, generalize the categorical equivalence $\mathbf{Grp} \simeq \mathbf{Ptd}^{conn}_{\leq 1}$ to 2-dimensional groups.

   Result: *biequivalence between coherent* 2*-groups and pointed connected* 2*-types.*

3. Find another classification of pointed connected 2-types by extending a classical bijection to the higher-type setting.

   Result: *type equivalence between pointed connected* 2*-types and Sính triples (a kind of cohomology data).*

   Everything is formalized in (non-cubical) Agda.[1]

4. Open problems

[1] `https://github.com/PHart3/2-groups-agda`

Motivation

- 2-groups and pointed connected 2-truncated spaces have rich histories in algebraic topology, forming two sides within the *homotopy hypothesis*.
- The biequivalence has been suggested previously in both the classical and the type-theoretic setting.
- The equivalence with Sính triples (named after Hoàng Xuân Sính) dates back to MacLane and Whitehead and gives a strict algebraic model for the spaces in question.
- Homotopy type theory lets us prove these two algebraic classifications in a constructive system with general semantics.

Dependent type theory

Martin-Löf type theory (MLTT) takes *types* and *type inhabitation* as primitive notions.

We try to prove judgments of the form

$$\Gamma \vdash t : X \qquad \textit{t has type X in context } \Gamma$$

Types can depend on terms: $P(x)$ *is a type depending on a term* x.

Fundamental type formers in MLTT:

- $(x : X) \to P(x)$ *for all* x *in* X, *we have an element of* $P(x)$.
- *The type* $(x : X) \times P(x)$ *of pairs* (x, b) *with* $b : P(x)$.
- For all $x, y : X$, the *identity type* $x = y$, inductively generated by reflexivity $\mathrm{refl}_x : x = x$. (Elements are called *paths*.)

Homotopy type theory (HoTT)

HoTT is an axiomatic extension of MLTT.

- Types are interpreted as *homotopy types* (as spaces up to homotopy equivalence).

  interpretation in all $(\infty, 1)$*-toposes (Shulman)*

- **Univalence axiom**

  Extensionality principle: Identity of types (in a type universe) is the same as equivalence of types.

- **Higher inductive types (HITs)**

  Generalize inductive types by allowing generators of id types.

  Let us directly build and study interesting spaces.

A critical notion

An *n-**type***, or *n-truncated type*, is a type whose iterated identity types are trivial above level n.

Examples

- A type X is a 0-type, or a *set*, if each of its loop spaces $x = x$ contains only refl_x.
- A type X is a 1-*type* if all of its loop spaces are sets.
- A type X is a 2-*type* if all of its loop spaces are 1-types.

We have a HIT $\|-\|_n$ that truncates any type to an n-type.

A type X is *n-connected* if $\|X\|_n \simeq \mathbf{1}$.

First classification: the homotopy hypothesis

Our first algebraic classification of pointed connected 2-types fits into a layered description of spaces as groupoids.

The homotopy hypothesis (Buchholtz–van Doorn–Rijke)

Idea: the *axiomatic* versus *internal* notion of higher groups

Groupoids	**Spaces**
group	pointed connected 1-type
abelian group	pointed n-connected $(n+1)$-type, $n > 0$
2-group	pointed connected 2-type
braided 2-group	pointed 1-connected 3-type
symmetric 2-group	pointed n-connected $(n+2)$-type, $n > 1$
$\vdots$	$\vdots$

The 1-dimensional case

1. The *delooping* of a group G (Licata and Finster):

   the 1-truncated HIT $K(G,1)$ generated by

   - a point base : $K(G,1)$
   - a homomorphism loop : $G \to \Omega(K(G,1), \mathsf{base})$:

   $$\begin{aligned} \mathsf{loop}_0 &: G \to \mathsf{base} = \mathsf{base} \\ \mathsf{loop}_1 &: (x, y : G) \to \mathsf{loop}_0(x \otimes y) = \mathsf{loop}_0(x) \cdot \mathsf{loop}_0(y) \end{aligned}$$

   Theorem: loop is an isomorphism.

2. Equivalence of categories (Buchholtz, van Doorn, and Rijke):[2]

$$\mathbf{Grp} \underset{\Omega}{\overset{K(-,1)}{\rightleftarrows}} \mathbf{Ptd}^{conn}_{\leq 1}$$

[2] They also handle the case for abelian groups.

2-groups

A *(coherent) 2-group* (Baez and Lauda) is a 1-type G with

- a binary operation $\otimes : G \to G \to G$, called the *tensor product*
- a neutral element id
- a right unitor ρ, a left unitor λ, and an associator α for $\otimes$

$$e.g., \quad \alpha \ : \ (x, y, z : X) \to (x \otimes y) \otimes z = x \otimes (y \otimes z)$$

- a triangle identity and a pentagon identity
- an inverse operation $(-)^{-1} : G \to G$
- paths $\mathrm{linv}_x : x^{-1} \otimes x = \mathrm{id}$ and $\mathrm{rinv}_x : x \otimes x^{-1} = \mathrm{id}$ for each $x : G$ such that linv and rinv satisfy two zig-zag identities.

That is, a 2-group is a univalent monoidal groupoid where every object is an adjoint equivalence.

Example

For every pointed 2-type X, the loop space $\Omega(X)$ equipped with path concatenation is a 2-group.

A *2-group morphism* $G_1 \to G_2$ is a function $f_0 : G_1 \to G_2$ equipped with a family of paths $\mu_{x,y} : f_0(x) \otimes f_0(y) = f_0(x \otimes y)$ that respects the associator.

Note: This short notion of 2-group morphism is equivalent to the one that explicitly preserves all data.

We have a *bicategory* **2Grp**.

A *bicategory* extends a category with 2-cells (morphisms between morphisms) and appropirate coherence laws.[3]

[3] We only consider $(2, 1)$-categories whose 2-cells are paths. Such data forms a *locally univalent bicategory* in the sense of Ahrens et al.

Delooping a 2-group

Let G be a 2-group.

Construct its delooping by generalizing the delooping $K(-,1)$ of an ordinary group.

Form the 2-truncated HIT $\mathcal{K}_2(G)$ generated by

- a point base : $\mathcal{K}_2(G)$
- a morphism of 2-groups loop : $G \to \Omega(\mathcal{K}_2(G), \mathrm{base})$

Key feature: no path constructors for units or inverses.

makes induction on $\mathcal{K}_2(G)$ far more tractable

Theorem (via brute-force induction): loop is an isomorphism.

The biequivalence

For bicategories $\mathcal{C}$ and $\mathcal{D}$, a *(biadjoint) biequivalence between $\mathcal{C}$ and $\mathcal{D}$* is a pseudofunctor $F : \mathcal{C} \to \mathcal{D}$ together with

- a pseudofunctor[4] $G : \mathcal{D} \to \mathcal{C}$
- a pseudotransformation $\tau_1 : F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$ each of whose components is an adjoint equivalence in $\mathcal{D}$
- a pseudotransformation $\tau_2 : \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$ each of whose components is an adjoint equivalence in $\mathcal{C}$
- A *single* zig-zag law between τ_1 and τ_2.

Note: This notion of biequivalence is fully coherent.

Theorem

We have a biequivalence

$$\mathbf{2Grp} \underset{\Omega}{\overset{\mathcal{K}_2}{\rightleftarrows}} \mathbf{Ptd}_{\leq 2}^{conn}$$

[4] *Pseudo* means we're extending a 1-categorical notion to bicategories.

Second classification: Sính triples

We now see that a pointed connected 2-type X is determined by its loop space. But what if we only have $\pi_1(X) :\equiv \|\Omega(X)\|_0$?

$$\begin{tikzcd}
& \textit{Pointed connected 2-types} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\textit{Coherent 2-groups} & & \textit{Sính triples}
\end{tikzcd}$$

MacLane and Whitehead gave a bijection between homotopy classes of pointed connected 2-truncated spaces and *Sính triples*:

- a group G
- a G-module M, i.e., an abelian group acted on by G
- a group cohomology class $\kappa \in H^3(G, M)$

For each suitable space X, take $\pi_1(X)$'s canonical action on $\pi_2(X)$.

The *type* of pointed connected 2-types is not a set (or a 1-type). So, how do we promote this bijection to a type equivalence?

Translation to higher group theory:

$$\begin{aligned}
\text{group } G &\mapsto \text{pointed connected 1-type} \\
G\text{-module } M &\mapsto \text{function } M : G \to \mathbf{Ab} \\
H^n(G, M) &\mapsto \|(x : G) \to_* K(M(x), n)\|_0 \\
&\qquad (K(M(x), n) := \text{higher EM space})
\end{aligned}$$

For the desired type equivalence, we "untruncate" $H^n(G, M)$ to get the type of *cocycles* on G over M.

Theorem

We have an equivalence of types:

$$\mathbf{Ptd}^{conn}_{\leq 2} \simeq \left(G : \mathbf{Ptd}^{conn}_{\leq 1}\right) \times (M : G \to \mathbf{Ab}) \times ((x : G) \to_* K(M(x), 3))$$

In fact, the internal approach to higher group theory gives us a proof that works in every dimension $n > 1$:

$$\begin{gathered}\mathbf{Ptd}^{conn}_{\leq n} \\ \wr\| \\ \left(G : \mathbf{Ptd}^{conn}_{\leq n-1}\right) \times (M : G \to \mathbf{Ab}) \times ((x : G) \to_* K(M(x), n+1))\end{gathered}$$

Key step in the proof:

Definition

Let X be a pointed type and let $\mathcal{U}$ be a universe. The type of *X-torsors* is $TX :\equiv (Y : \mathcal{U}) \times \|Y\| \times ((y : Y) \to X \simeq_* (Y, y))$.

Theorem (Wärn)

If $(\tau : TX) \times \mathrm{pr}_1(\tau)$ is contractible, then TX is the unique connected delooping of X.

With this theorem, we can recast $K(M(x), n+1)$ as $T(K(M(x), n))$, enabling a number of helpful manipulations.

Recovering the classical bijection

MacLane and Whitehead's proof relies on choice.

Within HoTT, we recover a constructive version of the bijection by applying the set truncation.

Corollary

We have a bijection:

$$\left\|\mathbf{Ptd}^{conn}_{\leq 2}\right\|_0 \simeq \left\|\left(G : \mathbf{Ptd}^{conn}_{\leq 1}\right) \times (M : G \to \mathbf{Ab}) \times H^3(G, M)\right\|_0$$

Open problems

1. Can we promote the type equivalence between $\mathbf{Ptd}_{\leq 2}^{conn}$ and the untruncated Sính triples to a biequivalence?

2. Can we find a tractable way to verify the remaining two cases of the dimension-2 homotopy hypothesis within HoTT?

Open problems

1. Can we promote the type equivalence between $\mathbf{Ptd}^{conn}_{\leq 2}$ and the untruncated Sính triples to a biequivalence?

2. Can we find a tractable way to verify the remaining two cases of the dimension-2 homotopy hypothesis within HoTT?

Thanks!

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